SKOLIAD No. 76

Shawn Godin

Please send your solutions to the problems from this issue by *1 September, 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will print solutions to problems marked with an asterisk ($*$) only if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's problems are drawn from the county-wide and national mathematics competitions held by the Croatian Mathematical Society in 2003. Thanks to Željko Hanjš of the Croatian Mathematical Society for making these problems available.

Croatian Mathematical Society County-Wide Competition
Junior Level (Grade 1), April 4, 2003

1. The lengths of the sides of a triangle ABC are $a = b - \frac{r}{4}$, b, $c = b - \frac{r}{4}$, where r is the radius of the inscribed circle. Determine the lengths of the sides of this triangle as a function of r only.

2. If $a > 0$, determine which points (x, y) in the Cartesian plane satisfy the inequality

$$\big||x+a| - |y-a|\big| \;<\; a\,.$$

3. Find all integer solutions to the equation

$$4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 \;=\; 0\,.$$

4. How many four-digit positive integers divisible by 7 have the property that, when the first and last digits are interchanged, the result is a (not necessarily four-digit) positive integer divisible by 7?

Croatian Mathematical Society National Competition Junior Level (Grade 1), May 7-10, 2003

1. Consider a triangle ABC whose sides have lengths which are prime numbers. Prove that the area of the triangle cannot be an integer.

2. The product of the positive real numbers x, y, and z is equal to 1. If

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; x+y+z\,,$$

prove that

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \;\geq\; x^k+y^k+z^k$$

for every positive integer k.

3. Consider an isosceles triangle ABC with base length a whose two equal sides are of length b and whose altitude is of length v. If $\frac{a}{2}+v \geq b\sqrt{2}$, determine the angles of the triangle. Furthermore, if $b = 8\sqrt{2}$, calculate the area of the triangle.

4. How many divisors of the number 30^{2003} are not divisors of 20^{2000}?

Next, we give the official solutions to those problems **not** marked with an asterisk ($*$) from the Ninteenth W.J. Blundon Mathematics Contest that appeared [2003 : 261–262].

4. Find all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x}\,)^x$.

Solution.

Note that $(x\sqrt{x}\,)^x = \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x}$. Thus, the equation to be solved is

$$x^{x\sqrt{x}} \;=\; x^{\frac{3}{2}x}\,.$$

The equation is obviously satisfied if $x = 1$. If $x \neq 1$, then we must have

$$\begin{aligned} x\sqrt{x} &= \tfrac{3}{2}x\,, \\ 2x\sqrt{x} &= 3x\,, \\ 4x^3 &= 9x^2\,, \\ 4x^3-9x^2 &= 0\,, \\ x^2(4x-9) &= 0\,, \\ x &= \tfrac{9}{4}\,, \end{aligned}$$

since $x \neq 0$ because $x > 0$. Thus, the positive solutions are 1 and $\frac{9}{4}$.

5. Rationalize the denominator: $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

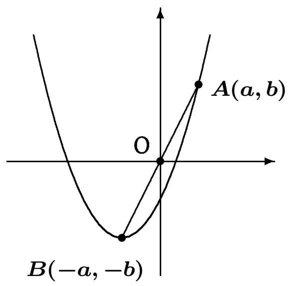
Solution.

$$\begin{aligned}
\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} &= \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} \cdot \frac{(\sqrt{2}+\sqrt{3})-\sqrt{6}}{(\sqrt{2}+\sqrt{3})-\sqrt{6}} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{5+2\sqrt{6}-6} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \cdot \frac{2\sqrt{6}+1}{2\sqrt{6}+1} \\
&= \frac{7\sqrt{2}+5\sqrt{3}-\sqrt{6}-12}{23}.
\end{aligned}$$

6. Points A and B are on the parabola $y = 2x^2 + 4x - 2$. The origin is the mid-point of the line segment joining A and B. Find the length of this line segment.

Solution.

Let A have coordinates (a, b). Then, since the origin is the mid-point of the line segment AB, point B has coordinates $(-a, -b)$. Since these points are on the parabola, we must have



$$b = 2a^2 + 4a - 2,$$

and

$$-b = 2(-a)^2 + 4(-a) - 2 = 2a^2 - 4a - 2.$$

Adding these two equations yields $4a^2 - 4 = 0$; that is, $a = \pm 1$. If $a = 1$, then

$$b = 2(1)^2 + 4(1) - 2 = 4.$$

If $a = -1$, then

$$b = 2(-1)^2 + 4(-1) - 2 = -4.$$

Thus, AB is the line segment joining $(1, 4)$ and $(-1, -4)$, which has length

$$\sqrt{[1-(-1)]^2 + [4-(-4)]^2} = \sqrt{4+64} = \sqrt{68} = 2\sqrt{17}.$$

7. If $\log_{125} 2 = a$ and $\log_9 25 = b$, find $\log_8 9$ in terms of a and b.

Solution.

We use $\log_8 9 = \dfrac{\ln 9}{\ln 8}$. Since $\log_9 25 = b$, we have

$$\begin{aligned} \frac{\ln 25}{\ln 9} &= b\,, \\ \ln 9 &= \frac{\ln 25}{b} = \frac{2\ln 5}{b}\,. \end{aligned}$$

Since $\log_{125} 2 = a$, we have

$$\begin{aligned} \frac{\ln 2}{\ln 125} &= a\,, \\ \ln 2 &= a\ln 125 = 3a\ln 5\,, \\ \ln 8 &= 3\ln 2 = 9a\ln 5\,. \end{aligned}$$

Therefore,

$$\log_8 9 = \frac{\ln 9}{\ln 8} = \frac{(2\ln 5)/b}{9a\ln 5} = \frac{2}{9ab}\,.$$

8. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Solution.

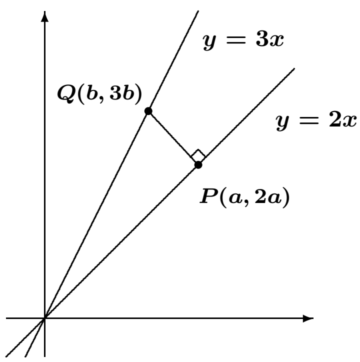
Let the coordinates of P be $(a, 2a)$ and the coordinates of Q be $(b, 3b)$. The slope of the line $y = 2x$ is 2. Hence, the slope of PQ is $-\frac{1}{2}$. Thus, we must have

$$\frac{3b - 2a}{b - a} = -\frac{1}{2}\,.$$

Therefore, $b = \frac{5}{7}a$. Since $PQ = 5$, we have

$$\begin{aligned} (b-a)^2 + (3b-2a)^2 &= 5^2\,, \\ \left(-\tfrac{2}{7}a\right)^2 + \left(\tfrac{1}{7}a\right)^2 &= 25\,, \\ \tfrac{5}{49}a^2 &= 25\,, \\ a &= 7\sqrt{5}\,. \end{aligned}$$

Thus, P has coordinates $(7\sqrt{5}, 14\sqrt{5})$.



9. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Solution.

The line is tangent to the circle if and only if the system

$$\begin{aligned} x + y &= a \\ x^2 + y^2 &= b \end{aligned}$$

has a unique solution. Solving, we get

$$\begin{aligned} x^2 + (a-x)^2 &= b\,, \\ \text{or} \quad 2x^2 - 2ax + a^2 - b &= 0\,. \end{aligned}$$

This equation has a unique solution if and only if the discriminant is zero. That is,

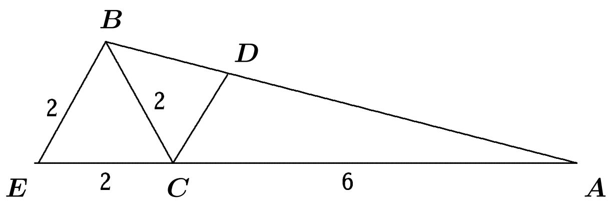
$$\begin{aligned} 4a^2 - 8(a^2 - b) &= 0\,, \\ 8b - 4a^2 &= 0\,, \\ 4(2b - a^2) &= 0\,, \\ a^2 &= 2b\,. \end{aligned}$$

Thus, the line is tangent to the circle when $a^2 = 2b$.

10. In $\triangle ABC$, we have $\angle ACB = 120$ degrees, $AC = 6$ and $BC = 2$. The internal bisector of $\angle ACB$ meets the side AB at the point D. Determine the length of the line segment CD.

Solution.

Draw BE parallel to DC meeting AC (extended) at E.



Then $\angle ACD = \angle AEB = 60°$ and $\angle DCB = \angle EBC = 60°$. Therefore, $\triangle BCE$ is an equilateral triangle. Also, $\triangle ADC$ is similar to $\triangle ABE$, since $BE \parallel CD$. Hence,

$$\begin{aligned} \frac{CD}{CA} &= \frac{EB}{EA}\,, \\ \frac{CD}{6} &= \frac{2}{8}\,, \\ CD &= \frac{3}{2}\,. \end{aligned}$$

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo) and Dan MacKinnon (Ottawa Carleton District School Board).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier septembre 2004**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

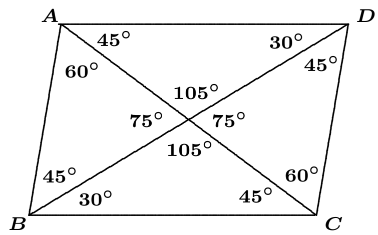
Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M132. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) Qui a-t-il de faux dans le dessin du parallélogramme ci-contre ? Les angles sont mesurés en degrés.

(b) Proposez une modification qui rendrait le dessin plausible. On ne doit pas modifier les segments issus de B.



M133. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans un pentagone $ABCDE$, chaque côté est parallèle à une diagonale. Montrer que le rapport d'une diagonale au côté parallèle correspondant est constant. En fait, cette constante est le nombre d'or. (Un tel pentagone est appelé *un pentagone d'or*.)

M134. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans un pentagone d'or $ABCDE$ (voir le problème précédent pour la définition), l'angle EAB est égal à l'angle BCD. Montrer que l'angle CDE est égal à l'angle DEA.

M135. *Proposé par l'Équipe de Mayhem.*

Trouver tous les nombres de deux chiffres avec exactement 8 diviseurs positifs.

M136. *Proposé par l'Équipe de Mayhem.*

Pour construire un nombre de cinq chiffres, on utilise chacun des chiffres 1, 4, 5, 7, et 8 une seule fois. Déterminer la somme de tous les différents nombres de cinq chiffres ainsi construits.

M137. *Proposé par Babis Stergiou, Lycio Psachnon Evias, Grèce.*

Soit a, b, $c > 0$, $a + b + c = 3$, et $abc = 1$.

(a) Montrer que $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$;

(b) Déduire de (a) que (ou sinon montrer que)

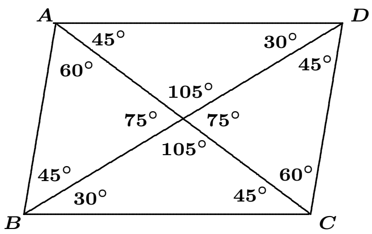
$$\frac{ab}{(a^2+b)(a+b^2)} + \frac{bc}{(b^2+c)(b+c^2)} + \frac{ca}{(c^2+a)(c+a^2)} \leq \frac{3}{4}\,.$$

. . . .

M132. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) What is wrong with the diagram of the parallelogram? The angle measures are in degrees.

(b) Suggest a modification that would make the diagram plausible. You are not allowed to modify the three lines through B.



M133. *Proposed by K.R.S. Sastry, Bangalore, India.*

In pentagon $ABCDE$, each side is parallel to a diagonal. Show that the ratio of a diagonal to the corresponding parallel side is constant. In fact, this constant is the golden ratio. (Such a pentagon is called a *golden pentagon*.)

M134. *Proposed by K.R.S. Sastry, Bangalore, India.*

In golden pentagon $ABCDE$ (see the preceding problem for the definition), we have $\angle EAB = \angle BCD$. Show that $\angle CDE = \angle DEA$.

M135. *Proposed by the Mayhem staff.*

Find all two-digit numbers with exactly 8 positive divisors.

M136. *Proposed by the Mayhem staff.*

The digits 1, 4, 5, 7, and 8 are each used once to form a five-digit number. Determine the sum of all such distinct five-digit numbers.

M137. *Proposed by Babis Stergiou, Lycio Psachnon Evias, Greece.*

Suppose a, b, $c > 0$, $a + b + c = 3$, and $abc = 1$.

(a) Prove that $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$.

(b) Hence, or otherwise, prove that

$$\frac{ab}{(a^2 + b)(a + b^2)} + \frac{bc}{(b^2 + c)(b + c^2)} + \frac{ca}{(c^2 + a)(c + a^2)} \leq \frac{3}{4}.$$

Mayhem Solutions

M69. *Proposed by the Mayhem Staff.*

A sequence of digits is formed by writing the digits from the natural numbers in the order that they appear. The sequence starts:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, \ldots.$$

What is the 2002^{nd} digit in the sequence?

Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

Writing the first 9 natural numbers requires 9 digits. Writing the next 90 numbers requires $90 \times 2 = 180$ digits. Thus, to write the numbers from 1 to 99 requires 189 digits. The next 9 groups of 100 natural numbers each require $100 \times 3 = 300$ digits. Clearly, only 6 of these groups are required. Once we have written the digits of the number 699, we will have used 1989 digits. Now, $2002 - 1989 = 13$ more digits are required. Therefore, the 2002^{nd} digit is the '7' from 704.

M70. *Proposed by the Mayhem Staff.*

What is the smallest positive multiple of 15 that is made up of only the digits 0, 4, and 7, each appearing the same number of times?

Solution by the Mayhem Staff.

Let n be the required positive integer. Clearly, $15|n$ implies both $5|n$ and $3|n$. Since $5|n$ and the only digits in n are 0, 4, and 7, it follows that n ends in the digit 0. Since $3|n$, the digital sum of n must be a multiple of 3, and thus, the number of repeats of each digit must be a multiple of 3. Since n is the *smallest* multiple of 15, we try the number of repeats equal to 3. Again, since we want the smallest positive integer, we need to have the smaller digits at the beginning of the integer. Thus, since leading zeroes are not allowed, $n = 400447770$.

M71. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Let $x = a + b - c$, $y = a + c - b$ and $z = b + c - a$, where a, b and c are prime numbers. Given that $x^2 = y$ and $\sqrt{z} - \sqrt{y}$ is the square of a prime number, determine all possible values for the product abc.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The unique solution of the simultaneous linear equations $a+b-c = x$, $a + c - b = y$, and $b + c - a = z$ is

$$(a, b, c) \;=\; \left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + z), \tfrac{1}{2}(y + z)\right) .$$

Setting $y = x^2$, we have

$$\begin{aligned} a &= \tfrac{1}{2}(x + x^2)\,, && (1)\\ b &= \tfrac{1}{2}(x + z)\,, && (2)\\ c &= \tfrac{1}{2}(x^2 + z)\,. && (3) \end{aligned}$$

From (1), we have $x = \dfrac{-1 \pm \sqrt{1 + 8a}}{2}$. Since x is an integer, we get $1 + 8a = T^2$, for some odd positive integer T. Hence, $2a = \dfrac{T-1}{2} \cdot \dfrac{T+1}{2}$. Therefore, since a is prime, we have $\dfrac{T-1}{2} = 2$ and $\dfrac{T+1}{2} = a$, from which we conclude that $T = 5$ and $a = 3$. Then, using (1), $x = 2$ or $x = -3$.

If $x = 2$, then $y = 4$ and $\sqrt{z} - 2 = p^2$, for some prime p. Thus, $z = \left(p^2 + 2\right)^2$. From (2), we see that z is even. Hence, $p = 2$ and $z = 36$. Substituting into (3), we get $c = 20$, which contradicts the fact that c is prime.

Therefore, $x = -3$. Then $y = 9$ and $z = \left(p^2 + 3\right)^2$, for some prime p. Now, (2) and (3) imply that z is odd. This happens only if $p = 2$, which gives $z = 49$. Then $b = 23$, and $c = 29$.

Then $abc = (3)(23)(29) = 2001$.

M72. *Proposed by J. Walter Lynch, Athens, GA, USA.*

You have a cup of coffee and a cup of tea. The cups are identical and each contains the same amount of liquid as the other. You take a teaspoon full of coffee out of the coffee cup and put it into the teacup. You then take a teaspoon full of the mixture out of the teacup and put it into the coffee cup. Which is greater, the amount of coffee in the teacup, the amount of tea in the coffee cup, or are they the same?

Solution by the Mayhem Staff.

Exactly 1 teaspoon of liquid was transferred in each direction; thus, the volume of liquid in each cup is the same both before and after the exchange. Suppose the amount of coffee in the teacup is greater than the amount of tea in the coffee cup. Then, since the total amount of coffee equals the total amount of tea, the remaining amount of coffee in the coffee cup is less than the remaining amount of tea in the teacup. This implies that the amount of liquid in the coffee cup is less than that in the teacup, which contradicts the statement in the first sentence.

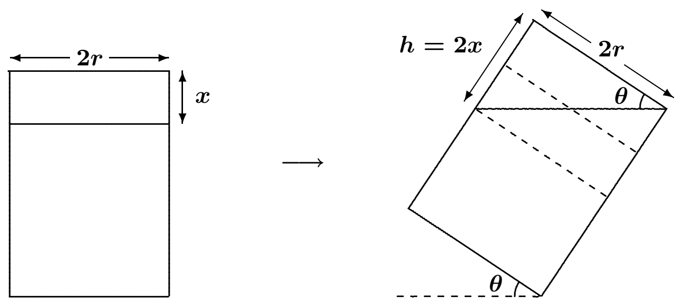
Similarly the supposition that the amount of coffee in the teacup is less than the amount of tea in the coffee cup leads to a contradiction.

Thus, these two amounts must be equal.

M73. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A right circular cylinder with radius r and height h contains a liquid to within x of the top of the cylinder. Find the angle through which the cylinder must be tilted in order for the liquid to start to pour out. (Assume that there is enough liquid in the cylinder so that the surface of the liquid does not intersect the bottom of the cylinder before the liquid starts to pour out.)

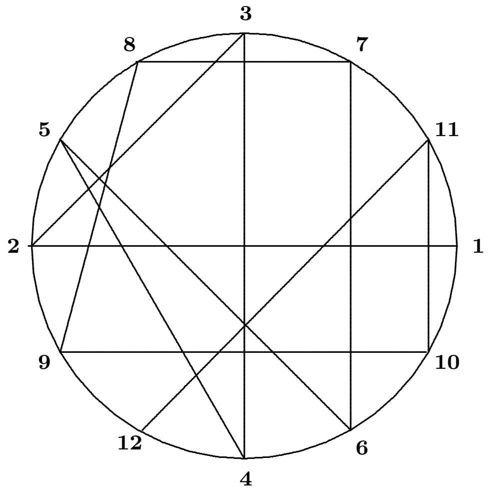
Solution by the Mayhem Staff.



As the cylinder is tilted, the volume of the unfilled space inside the cylinder remains constant as long as no liquid spills over. Consider the cylinder tilted to the point where liquid is just starting to spill (as in the figure above). Let θ denote the angle of tilt at this point. The liquid surface is now an ellipse. Draw a circle around the cylinder from the point on this ellipse which is diametrically opposite the lip of the cylinder. This circle marks off a cylinder of height h at the top of the original cylinder. The surface ellipse of the liquid divides this second cylinder into two congruent 'wedges', one of which is the unfilled space. Therefore, the volume of the entire second cylinder is twice that of the unfilled space. It follows that $h = 2x$. Hence, the angle θ is given by $\tan\theta = \dfrac{2x}{2r} = \dfrac{x}{r}$; that is, $\theta = \arctan\left(\dfrac{x}{r}\right)$.

M74. *Proposed by the Mayhem Staff.*

A circle has 12 equally spaced points placed on its circumference. How many ways can the numbers 1 through 12 be assigned to the points so that if the points 1 through 12 are connected with line segments, in order, the segments do not cross? An example of a **bad** arrangement is illustrated below.



Solution by the Mayhem Staff.

We will use the fact that there is at least one chord drawn to or from every point. We can place the label '1' at any of the 12 points. However, the label '2' is now forced to go at one of the two points adjacent to the point labelled '1', since, if we skip past an adjacent point, a chord to or from this adjacent point will intersect the chord joining the points that we have labelled '1' and '2'. Similarly, the label '3' can only be placed at one of the two points adjacent to the '1–2' block. There is a choice of two possible points for each successive label until the label '12', for which there is only one remaining point. Thus, there are $12\left(2^{10}\right) = 12\,288$ labellings.

M75. *Proposed by the Mayhem Staff.*

The increasing sequence 1, 5, 6, 25, 26, 30, 31, 125, 126, ... consists of positive integers that can be formed by adding distinct powers of 5. What is the 75^{th} integer in the sequence?

Solution by the Mayhem Staff.

We write the sequence in base 5 as 1, 10, 11, 100, 101, 110, ... If, instead of considering this sequence in base 5, we now consider it as a sequence of binary numbers, we note that the number in position n is n. Thus, since 75 may be written in binary as 1001011, the required number is $5^6 + 5^3 + 5 + 1 = 15\,756$.

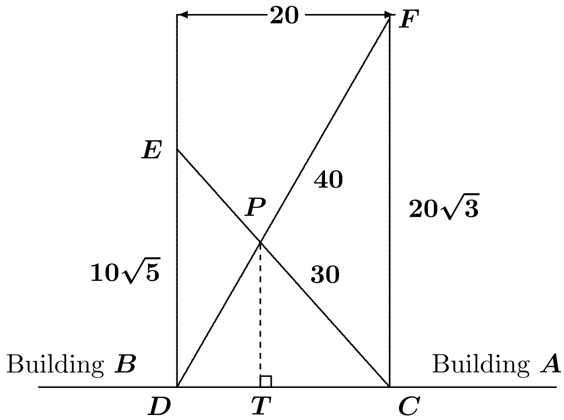
M76. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Two buildings A and B are twenty feet apart. A ladder thirty feet long has its lower end at the base of building A and its upper end against building B. Another ladder forty feet long has its lower end at the base of building B and its upper end against building A.

How high above the ground is the point where the ladders intersect?

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let CE and DF be the ladders, as shown in the diagram below. Let P be the point of intersection of the ladders, and let T be the point on the ground directly below P.



Applying the Pythagorean Theorem, we obtain $DE = 10\sqrt{5}$ and $CF = 20\sqrt{3}$. Since $\triangle CTP$ is similar to $\triangle CDE$ and $\triangle DTP$ is similar to $\triangle DCF$, we have the equations

$$\frac{CT}{PT} = \frac{2}{\sqrt{5}} \quad\text{and}\quad \frac{TD}{PT} = \frac{1}{\sqrt{3}} .$$

Adding these equations and noting that $CT + TD = 20$ yields

$$\frac{20}{PT} = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3}} ,$$

from which we see that the unique solution is

$$PT = \frac{20\sqrt{15}}{\sqrt{5}+2\sqrt{3}} = \frac{20(6\sqrt{5}-5\sqrt{3})}{7} .$$

Also solved by Andrew Mao, A.B. Lucas Secondary School, London, ON; and Yifei Chen, West Windsor Plainsboro High School North, Plainsboro, NJ, USA.

Pólya's Paragon

What's the difference?

Shawn Godin

"Little Johnnie encounters the following list of numbers 12, 49, 62, 57, 40, 17, $\ldots$. *What is the next number in the list?"*

Questions like this have been seen in mathematics text books and on mathematics contests over the years. Is there a general way to attack them to find the next few terms?

It turns out that there is a powerful method called *finite differences* that deals with sequences quite well. The main idea is to find the differences between consecutive terms and look for a pattern. Many mathematical functions reveal their structure under this method. Let's see a couple of examples in action.

Example 1: Find the next term in the sequence 1, 3, 5, 7, 9, $\ldots$.

Solution: This example is trivial, but it reveals our general technique. If we call the terms t_1, t_2, t_3, t_4, t_5, $\ldots$ and the differences $d_1 = t_2 - t_1$, $d_2 = t_3 - t_2$, etc., we get the following table:

$$\begin{array}{c|ccccccccc} t_n & 1 & & 3 & & 5 & & 7 & & 9 \\ d_n & & 2 & & 2 & & 2 & & 2 & \end{array}$$

In case we didn't see the pattern in the original sequence, the sequence of differences is "easier". We can produce the next term in the original sequence by realizing that $t_6 - t_5 = d_5 = 2$ (since all the differences are 2), and therefore,

$$t_6 = 2 + t_5 = 2 + 9 = 11\,.$$

Technically, the differences that we produced are called *first differences*. If we rename our differences ${}_1d_1$, ${}_1d_2$, ${}_1d_3$, ${}_1d_4$, ${}_1d_5$, $\ldots$, then we can define the *second differences* as ${}_2d_1 = {}_1d_2 - {}_1d_1$, ${}_2d_2 = {}_1d_3 - {}_1d_2$, etc. These will be useful in the next problem.

Example 2: Find the next term in the sequence 1, 4, 9, 16, 25, $\ldots$.

Solution: Again, we have a sequence that is easy to recognize. This time the first differences are not the same; hence, we will continue to the second differences.

$$\begin{array}{c|ccccccccc} t_n & 1 & & 4 & & 9 & & 16 & & 25 \\ {}_1d_n & & 3 & & 5 & & 7 & & 9 & \\ {}_2d_n & & & 2 & & 2 & & 2 & & \end{array}$$

If we didn't see a pattern in the first differences, we can easily see one in the second differences. The next second difference must be ${}_2d_4 = 2$, which gives the next first difference as ${}_1d_5 = 9 + 2 = 11$, and then the next term of the sequence is $t_6 = 25 + 11 = 36$.

At this time a pattern is emerging. Notice that when the sequence can be modelled by a linear function, the first differences are constant. Similarly, when the sequence can be modelled by a quadratic, the second differences are constant. This pattern continues:

Theorem. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the k^{th} differences are constant.*

We can see the proof of this by looking at a lemma first.

Lemma. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the first differences $\{{}_1d_n\}$ can be modelled by a polynomial of degree $k - 1$.*

Proof: We will only sketch the main idea (try to construct your own proof). Suppose we have a sequence $\{t_n\}$ where $t_n = n^k$. Then

$${}_1d_n \;=\; t_{n+1} - t_n \;=\; (n+1)^k - n^k\,.$$

Using the Binomial Theorem, we can see that ${}_1d_n$ is of degree $k - 1$.

Using the lemma, you can prove the theorem by induction. With a little experimentation, you will also see that the difference that is constant can be used to determine the coefficient of the highest degree term. See if you can discover the connection, and provide a proof. With this new theorem, you should be set to attack the original sequence.

We will return to this topic next issue and expand on it. Here are some problems to keep you busy in the meantime (homework, one might say).

Examine the differences for the following sequences, and see if you can make any predictions.

1. $a_n = 2^n$.

2. $b_n = 5^n$.

3. $c_n = 2 \times 3^n$.

4. $1, 1, 2, 3, 5, 8, 13, \ldots$. (This is the *Fibonacci sequence*. Each term is the sum of the previous two terms.)

It should be noted, that any sequence like the one with which we started can be continued in **any** *way. That is, we can pick any number to go next and find a polynomial that will match those numbers. See the article on Lagrange Interpolation from the Skoliad Corner* [2001 : 386–388].

The Birthday Problem Revisited

Sandra M. Pulver

Problems about birthday probabilities are among the most interesting problems in probability due to their surprising answers, and they are more germane to the beginning probability student than many of the other problems in elementary probability texts. We will discuss three of the many birthday problems in existence.

In these problems, February 29th is ignored as a possible birthday and the other 365 days are treated as equally likely.

The Classical Birthday Problem. *What is the minimum number of people so that the probability of two or more of them having the same birthday exceeds one half?*

We need to find the probability that, among n people, at least two have a common birthday. We will solve the complementary problem of finding the probability that *no two people have the same birthday*.

Since each person has 365 possible birthdays, the number of possible ways for n people to have birthdays is 365^n. The number of ways for n people to have no matching birthdays is

$$365 \times 364 \times \cdots \times (365 - n + 1)\,,$$

because there are 365 possible birthdays for the first person, 364 possible *different* birthdays for the second person, 363 for the third person, and so on. The last person can have a birthday different from the first $(n-1)$ people in $365 - (n - 1)$ ways. Thus, the probability that no two of the n people have the same birthday is

$$\frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

Now the probability P_n of at least one matching pair of birthdays among the n people is

$$P_n \;=\; 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

According to our objective, we have to find the smallest n such that $P_n > \frac{1}{2}$. The following table shows that the answer is 23.

n	5	10	15	20	21	22	23	24	25
P_n	0.027	0.117	0.253	0.411	0.444	0.476	0.507	0.538	0.569

The Birthmate Problem. *What is the smallest number of strangers you need to meet in order to have at least a 50% chance of finding one whose birthday is the same as yours?*

This time, we need to find the probability that, among n strangers, at least one has the same birthday as yours. We again consider the complementary problem of finding the probability that no birthdays are the same as yours.

First of all, the probability that some birthday is different from yours is $\frac{364}{365}$, or $\frac{N-1}{N}$, if we set $N = 365$. Since we have n strangers, the probability that nobody among them has your birthday is $\left(\frac{N-1}{N}\right)^n$. Then the probability P_n that at least one person's birthday matches yours is

$$P_n = 1 - \left(\frac{N-1}{N}\right)^n = 1 - \left(1 - \frac{1}{N}\right)^n .$$

As N increases, the value of $\left(1 - \frac{1}{N}\right)^N$ approaches e^{-1}. We therefore have the approximation

$$\left(1 - \frac{1}{N}\right)^N \approx e^{-1} .$$

Taking the N^{th} root of both sides, we obtain $1 - \frac{1}{N} \approx e^{-\frac{1}{N}}$. Now,

$$\left(1 - \frac{1}{N}\right)^n \approx \left(e^{-\frac{1}{N}}\right)^n = e^{-\frac{n}{N}} .$$

Hence, $P_n \approx 1 - e^{-\frac{n}{N}}$.

We want $P_n \geq \frac{1}{2}$, which implies $e^{-\frac{n}{N}} \leq \frac{1}{2}$. Taking natural logarithms on both sides, we have

$$-\frac{n}{N} \approx \ln\left(\tfrac{1}{2}\right) \approx -0.693 ,$$

or $n \approx 0.693N$. Since $N = 365$, we have

$$n \approx 252.945 .$$

Since n must be an integer, it must be 253. Thus, we need to ask at least 253 strangers to have at least a 50% chance that one of their birthdays will coincide with our own.

Birthmate Problem for a Group. *If n people meet by chance, what is the probability that they all have the same birthday?*

Observe that the probability that any given day is the birthday for a given person is $\frac{1}{365}$. Therefore, the probability that n persons all have a particular birthday is $\frac{1}{365^n}$. Now, adding over all 365 possible birthdays, the probability that all n persons have the same birthday is

$$365\left(\frac{1}{365^n}\right) = \frac{1}{365^{n-1}}\,.$$

To this point we have assumed that we were ignoring leap years. This allowed us to simplify the calculations. Readers may wish to reconsider the previous examples without the assumption. In this final example, if we were working with a leap year instead, the probability would change to $\frac{1}{366^{n-1}}$. In order to consider both possibilities, we look at a 4–year cycle, in which case the probability can be seen to be $(1 + 365 \times 4^n)/1461^n$. All three of these probabilities are computed in the following table for selected values of n.

n	2	10	20
non-leap year	0.002740	8.697×10^{-24}	2.072×10^{-49}
leap year	0.002732	8.485×10^{-24}	1.967×10^{-49}
overall	0.002736	8.638×10^{-24}	2.044×10^{-49}

Therefore, for all n people to have the same birthday is very unlikely.

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THE OLYMPIAD CORNER

No. 236

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As a first set of Olympiad problems, we give the Hungary-Israel Binational Mathematical Competition 2001. Thanks go to Chris Small, Canadian Team Leader to the 46^{th} IMO, for collecting them.

HUNGARY-ISRAEL BINATIONAL MATHEMATICAL COMPETITION 2001

Individual Competiton

1. Find positive integers x, y, z such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

2. Points A, B, C, D lie on the line ℓ, in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real x,

$$f(f(x)) \;=\; f(x) + x\,.$$

4. Let $P(x) = x^3 - 3x + 1$. Find the polynomial Q whose roots are the fifth power of the roots of P.

5. A triangle ABC is given. The mid-points of sides AC and AB are B_1 and C_1, respectively. The centre of the incircle of $\triangle ABC$ is I. The lines B_1I, B_2I meet the sides AB, AC at C_2, B_2, respectively. Given that the areas of $\triangle ABC$ and $\triangle AB_2C_2$ are equal, what is $\angle BAC$?

6. Given are 32 positive integers with a sum of 120, one of which is greater than 60. Prove that these integers can be divided into two disjoint subsets that have the same sum.

Team Competiton

In the following questions, G_n is a simple undirected graph with n vertices, K_n is the complete graph with n vertices, $K_{n,m}$ is the complete bipartite graph with m vertices in one of the two partite sets and n vertices in the other, and C_n is a circuit with n vertices. The number of edges in the graph G_n is denoted $e(G_n)$.

1. The edges of K_n, $n \geq 3$, are coloured with n colours, and every colour appears at least once. Prove that there is a triangle whose sides are coloured with 3 different colours.

2. An integer $n \geq 5$ is given. If $e(G_n) \geq \frac{n^2}{4} + 2$, prove that there exist two triangles which have exactly one common vertex.

3. If $e(G_n) \geq \frac{n\sqrt{n}}{2} + \frac{n}{4}$, prove that G_n contains C_4.

4. (a) If G_n does not contain $K_{2,3}$, prove that $e(G_n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + n$.

(b) Given $n \geq 16$ distinct points $P_1, P_2, \ldots, P_n$ in the plane, prove that at most $n\sqrt{n}$ of the segments P_iP_j have unit length.

5. (a) Let p be a prime. Consider the graph whose vertices are the ordered pairs (x, y) with $x, y \in \{0, 1, 2, \ldots, p-1\}$, and whose edges join vertices (x, y) and (x', y') if and only if $xx' + yy' \equiv 1 \pmod{p}$. Prove that this graph does not contain C_4.

(b) Prove that for infinitely many values of n, there is a graph G_n that does not contain C_4 and satisfies $e(G_n) \geq \frac{n\sqrt{n}}{2} - n$.

Next we turn to the problems of the Second Hong Kong (China) Mathematical Olympiad written December 1999. Thanks again go to Chris Small, Canadian Team Leader to the 46th IMO, for forwarding them to us.

SECOND HONG KONG (CHINA) MATHEMATICAL OLYMPIAD

December 1999

Time: 3 hours

1. [5 marks] Determine all positive rational numbers $r \neq 1$ such that $r^{1/(r-1)}$ is rational.

2. [10 marks] Let I and O be the incentre and circumcentre, respectively, of $\triangle ABC$. Assume $\triangle ABC$ is not equilateral (so that $I \neq O$). Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$.

3. [10 marks] Students have taken a test in each of n subjects ($n \geq 3$). It is known that, for any subject, exactly three students got the best score in the subject, and for any two subjects, exactly one student got the best score in both of the subjects. Determine the smallest n so that the above conditions imply that exactly one student got the best score in all n subjects.

4. [10 marks] Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$,

$$f(x + yf(x)) = f(x) + xf(y) .$$

A third set of problems for your puzzling pleasure are those of the 17th Balkan Mathematical Olympiad, written in May, 2000. Thanks again go to Chris Small.

17th BALKAN MATHEMATICAL OLYMPIAD
Chisinau, Republic of Moldova
May 5, 2000 — Time: 4.5 hours

Each problem is worth 10 points.

1. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(xf(x) + f(y)) = \left(f(x)\right)^2 + y ,$$

for any real numbers x and y.

2. Let ABC be a non-isosceles acute triangle, and let E be an interior point of the median AD, with D on BC. Let F be the orthogonal projection of E onto the line BC. Let M be an interior point of the segment EF, and let N and P be the orthogonal projections of M onto the lines AC and AB, respectively. Prove that the bisectors of angles PMN and PEN are parallel.

3. Find the maximal number of rectangles of size $1 \times 10\sqrt{2}$ which can be cut off from a rectangle of size 50×90 using cuts parallel to the edges of the initial 50×90 rectangle.

4. We say that a positive integer r is a *power* if it has the form $r = t^s$, for some integers $t \geq 2$ and $s \geq 2$. Show that, for any positive integer n, there exists a set A of n positive integers which satisfies the following conditions:

(i) Every element of A is a power.

(ii) For any k elements $r_1, r_2, \ldots, r_k$ from A (where $2 \leq k \leq n$), the number $\dfrac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Now we turn to our readers for solutions to problems of the Hungary-Israel Mathematical Competition 1999 given [2001: 421–422].

1. Let $f(x)$ be a polynomial whose degree is at least 2. Define the sequence $g_i(x)$ by: $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n = 1, 2, \ldots$. Let r_n be the average of the roots of $g_n(x)$. It is given that $r_{19} = 99$. Find r_{99}.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution, modified slightly by the editor.

Let $f(x) = \sum\limits_{i=0}^{p} c_i x^i$, where $p \geq 2$. Since $g_n = \underbrace{f \circ f \circ \ldots \circ f}_{n}$, we easily deduce that g_n is a polynomial of degree p^n. In $g_n(x)$, let α_n and β_n be the coefficients of x^{p^n} and x^{p^n-1}, respectively. Since g_n has p^n roots,

$$r_n \;=\; -\frac{\beta_n}{p^n \alpha_n}\,. \tag{1}$$

For each $n \geq 1$,

$$g_{n+1}(x) \;=\; f\big(g_n(x)\big) \;=\; \sum_{i=0}^{p} c_i \big(g_n(x)\big)^i \;=\; c_p\big(g_n(x)\big)^p + \sum_{i=0}^{p-1} c_i\big(g_n(x)\big)^i\,.$$

The degree of $\sum\limits_{i=0}^{p-1} c_i\big(g_n(x)\big)^i$ is at most $(p-1)p^n$. Therefore, terms of orders p^{n+1} and $p^{n+1} - 1$ appear only in the expansion of $c_p\big(g_n(x)\big)^p$. We have

$$c_p\big(g_n(x)\big)^p \;=\; c_p\left(\alpha_n x^{p^n} + \beta_n x^{p^n-1} + h_n(x)\right)^p\,,$$

where $\deg(h_n) \leq p^n - 2$. Applying the Binomial Theorem, we deduce that

$$\alpha_{n+1} \;=\; c_p \alpha_n^p \quad \text{and} \quad \beta_{n+1} \;=\; p c_p \alpha_n^{p-1} \beta_n\,.$$

Then, using (1), we have

$$r_{n+1} \;=\; -\frac{\beta_{n+1}}{p^{n+1}\alpha_{n+1}} \;=\; -\frac{p c_p \alpha_n^{p-1}\beta_n}{p^{n+1} c_p \alpha_n^p} \;=\; -\frac{\beta_n}{p^n \alpha_n} \;=\; r_n\,.$$

By induction, $r_n = r_1$ for all $n \in \mathbb{N}$. Thus, $r_{99} = r_{19} = 99$.

2. A set of $2n+1$ lines in a plane is drawn. No two of them are parallel, and no three pass through one point. Every three of these lines form a non-right triangle. Determine the maximal number of acute-angled triangles that can be formed.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let ℓ_1, ℓ_2, ..., ℓ_{2n+1} be the lines, and let M_{ij} be the common point of ℓ_i and ℓ_j (for $i \neq j$). Consider that we are in the complex plane $(O, \overrightarrow{u}, \overrightarrow{v})$.

Let L be an arbitrary line, distinct from and not parallel to any of the ℓ_i's, and such that none of the M_{ij}'s belongs to L. Let α_i be the angle between the lines L and $\ell_i \pmod{\pi}$.

Then, the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k is $M_{ij}M_{jk}M_{ki}$, and

$$2(\overrightarrow{M_{ij}M_{ik}}, \overrightarrow{M_{ij}M_{kj}}) \;=\; 2(L, \ell_i) + 2(\ell_j, L) \;=\; 2(\alpha_i - \alpha_j) \pmod{2\pi}\ .$$

Let P_i be the point on the unit circle Γ with center O such that

$$(\overrightarrow{u}, \overrightarrow{OP_i}) \;=\; 2\alpha_i \pmod{2\pi}\ .$$

Then, for i, j, k pairwise distinct, we have

$$\begin{aligned} 2(\overrightarrow{P_kP_j}, \overrightarrow{P_kP_i}) &= (\overrightarrow{OP_j}, \overrightarrow{OP_i}) \;=\; 2(\alpha_i - \alpha_j) \\ &= 2(\overrightarrow{M_{ij}M_{ik}}, \overrightarrow{M_{ij}M_{kj}}) \pmod{2\pi}\ . \end{aligned}$$

Thus, $\angle P_iP_kP_j = \angle M_{ik}M_{ij}M_{kj}$. It follows that $\angle M_{ik}M_{ij}M_{kj}$ is acute if and only if $\angle P_iP_kP_j$ is acute, and $\angle M_{ik}M_{ij}M_{kj}$ is obtuse if and only if $\angle P_iP_kP_j$ is obtuse (since there is no right triangle).

Moreover, we note that if points P_1, P_2, ..., P_{2n+1} are given on Γ, then we may find some lines ℓ_1, ℓ_2, ..., ℓ_{2n+1} such that the construction above leads to the given P_i's.

Therefore, the problem is equivalent to finding the maximum number of acute triangles formed by the P_i's. That is, it is equivalent to finding the minimum number of obtuse triangles formed by these points. But the triangle $P_iP_kP_j$ is obtuse if and only if P_i, P_j, and P_k belong to one semicircle defined on Γ.

Let $i \in \{1, 2, \ldots, 2n+1\}$ be fixed. Suppose that there are d_i points on one side of the diameter with endpoint P_i. Then there are $2n - d_i$ points on the other side of it (since there is no right triangle). The number of obtuse triangles with vertex P_i and an acute angle at P_i is

$$\begin{aligned} N_i &= \binom{d_i}{2} + \binom{2n-d_i}{2} \;=\; d_i^2 + 2n^2 - 2nd_i - n \\ &= (d_i - n)^2 + n^2 - n \;\geq\; n^2 - n\,, \end{aligned}$$

with equality if and only if $d_i = n$.

Summing over i, we count each obtuse triangle exactly twice. Thus, the number of obtuse triangles is

$$N \;=\; \frac{1}{2}\sum_{i=1}^{2n+1} N_i \;\geq\; \frac{(2n+1)n(n-1)}{2}\,.$$

Since there are exactly $\binom{2n+1}{3} = \frac{(2n+1)n(2n-1)}{3}$ triangles formed by the P_i's, it follows that the number of acute triangles is at most

$$\frac{(2n+1)n(2n-1)}{3} - \frac{(2n+1)n(n-1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

This value is achieved if, for example, we choose the P_i's as the vertices of a regular $(2n+1)$–gon inscribed in Γ.

Thus, the maximal number of acute triangles is $\frac{n(n+1)(2n+1)}{6}$.

3. Find all the functions f from the set of rational numbers to the set of real numbers such that for all rational x, y,

$$f(x+y) = f(x)f(y) - f(xy) + 1.$$

Solved by Michel Bataille, Rouen, France; and Christopher J. Bradley, Bristol, UK. We give Bataille's write-up.

The functions $x \mapsto 1$ and $x \mapsto x+1$ are clearly solutions. We now show that there is no other solution.

Suppose $f : \mathbb{Q} \to \mathbb{R}$ satisfies

$$f(x+y) = f(x)f(y) - f(xy) + 1 \tag{1}$$

for all x, $y \in \mathbb{Q}$. Taking $x = y = 0$, we get $f(0) = 1$. Then, taking $y = -x$ (for any $x \in \mathbb{Q}$), we get

$$f(x)f(-x) = f(-x^2). \tag{2}$$

From (2), $f(-1) = 0$ or $f(1) = 1$.

If $f(1) = 1$, then, using (1),

$$f(x) = f\big((x-1)+1\big) = f(x-1)f(1) - f(x-1) + 1 = 1,$$

for all $x \in \mathbb{Q}$. Hence, f is the constant function $x \mapsto 1$.

Suppose now that $f(-1) = 0$, and let $a = f(1)$. Taking $x = y = -1$ in (1) gives $f(-2) = \big(f(-1)\big)^2 - f(1) + 1 = 1 - a$. Then, taking $x = 1$ and $y = -2$ in (1), we get

$$\begin{aligned} f(-1) &= f(1)f(-2) - f(-2) + 1, \\ 0 &= a(1-a) - (1-a) + 1, \\ &= a(2-a). \end{aligned}$$

Therefore, $a = 0$ or $a = 2$.

If $a = 0$ (that is, $f(1) = 0$), then, from (1),

$$\begin{aligned} f(x) &= f(x-1)f(1) - f(x-1) + 1 = 1 - f(x-1) \\ &= 1 - \big(f(x)f(-1) - f(-x) + 1\big) = f(-x), \end{aligned}$$

showing that f is even. Then (2) gives $\left(f(\frac{1}{2})\right)^2 = f\left(\frac{1}{4}\right)$. It follows that

$$a \;=\; f\left(\tfrac{1}{2}+\tfrac{1}{2}\right) \;=\; \left(f\left(\tfrac{1}{2}\right)\right)^2 - f\left(\tfrac{1}{4}\right) + 1 \;=\; 1\,,$$

contradicting $a = 0$.

Thus, $f(1) = 2$. Using (1), we deduce that $f(x+1) = f(x) + 1$. By an easy induction, $f(x+n) = f(x) + n$ for all $x \in \mathbb{Q}$ and $n \in \mathbb{N}$. Recalling that $f(0) = 1$, we get $f(n) = n + 1$. Then, using (1) again,

$$\begin{aligned} f(x+n) &= f(x)f(n) - f(nx) + 1\,, \\ f(x) + n &= (n+1)f(x) - f(nx) + 1\,, \\ f(nx) &= nf(x) - n + 1\,. \end{aligned}$$

Now, let $r = m/n$, where m, $n \in \mathbb{N}$. Then

$$m + 1 \;=\; f(m) \;=\; f(nr) \;=\; nf(r) - n + 1\,.$$

Thus, $f(r) = \dfrac{m+n}{n} = 1 + r$. Moreover,

$$\begin{aligned} f(-r) &= f\big((1-r) + (-1)\big) \;=\; f(1-r)f(-1) - f(r-1) + 1 \\ &= -\big(f(r) - 1\big) + 1 \;=\; 2 - f(r) \;=\; 1 + (-r)\,. \end{aligned}$$

As a result, $f(x) = x + 1$ for all $x \in \mathbb{Q}$, and the proof is complete.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is almost equivalent to problem #3 of the Mathematical Olympiad in Bosnia and Herzegovina 1997 [2000 : 326]. From it, one can show that the two solutions are $f(x) = 1$ and $f(x) = x + 1$.

4. Let c be a positive integer. Define the following sequence:

$$a_1 \;=\; c, \quad a_{n+1} \;=\; ca_n + \sqrt{(c^2-1)(a_n^2-1)}\,, \quad n = 1, 2, \ldots.$$

Prove that all the terms a_n are positive integers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bataille's approach.

Since $c \geq 1$, we have $a_1 = c = \cosh(x)$ for some non-negative real number x. Induction now shows that $a_n = \cosh(nx)$ for all n. Indeed, if $a_n = \cosh(nx)$ for some n, then

$$\begin{aligned} a_{n+1} &= \cosh(x)\cdot\cosh(nx) + \sqrt{\big(\sinh(x)\big)^2\big(\sinh(nx)\big)^2} \\ &= \cosh(x)\cdot\cosh(nx) + \sinh(x)\cdot\sinh(nx) \\ &= \cosh\big((n+1)x\big)\,. \end{aligned}$$

It follows that $a_{n+2} + a_n = 2\cosh(x) \cdot \cosh\big((n+1)x\big) = 2ca_{n+1}$. Now, taking into account that $a_1 = c \in \mathbb{Z}$ and $a_2 = 2c^2 - 1 \in \mathbb{Z}$, and using the relation $a_{n+2} = 2ca_{n+1} - a_n$, an immediate induction shows that $a_n \in \mathbb{Z}$ for all n. The result follows, since $a_n = \cosh(nx) \geq 1$.

5. The function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{x+y+z}$$

is defined for every x, y, z such that $x+y+z \neq 0$. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

Solution by Christopher J. Bradley, Bristol, UK.

A solution is

$$(x_0, y_0, z_0) = (0.0009998, -0.0009998, 0.000001) .$$

Note that $x_0 + y_0 + z_0 = 0.000001$ and $x_0^2 + y_0^2 + z_0^2 = 0.0000019992\ldots$ Thus, $f(x_0, y_0, z_0) = 1.9992\ldots$, which lies between 1.999 and 2.

(This solution required only one "difficult" calculation, namely $(0.0009998)^2 = (0.001)^2(1-0.0002)^2$ —which is not all that difficult.)

Now we turn to solutions to problems of the 12th Korean Mathematical Olympiad, Final Round, given [2001 : 422–423].

1. Let R, r be the circumradius, and the inradius of $\triangle ABC$, respectively, and let R', r' be the circumradius and inradius of $\triangle A'B'C'$, respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the two triangles are similar.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

It is well known (see [2001 : 46]) that

$$\cos A + \cos B + \cos C = 1 + \frac{R}{r} .$$

Since $\dfrac{R}{r} = \dfrac{R'}{r'}$, we have

$$\cos A + \cos B + \cos C = \cos A' + \cos B' + \cos C' .$$

Since $C = C'$, we deduce that $\cos A + \cos B = \cos A' + \cos B'$; that is,

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2\cos\left(\frac{A'+B'}{2}\right)\cos\left(\frac{A'-B'}{2}\right) .$$

Now, note that $A + B = \pi - C = \pi - C' = A' + B'$. Thus,

$$\cos\left(\frac{A-B}{2}\right) \;=\; \cos\left(\frac{A'-B'}{2}\right)\,.$$

With no loss of generality, we may suppose that $A \geq B$ and $A' \geq B'$. Then $A - B = A' - B'$. Since we also have $A + B = A' + B'$, we deduce that $A = A'$ and $B = B'$. Then $\triangle ABC$ is similar to $\triangle A'B'C'$.

2. Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \;\leq\; \frac{n}{m}$ for all rational numbers n and m. Show that for all natural numbers k

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \;\leq\; \frac{k(k-1)}{2}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Christopher J. Bradley, Bristol, UK. We give Aassila's write-up.

For all natural numbers i, we have

$$\left|f(2^{i+1}) - f(2^i)\right| \;=\; \left|f(2^i + 2^i) - f(2^i)\right| \;\leq\; \frac{2^i}{2^i} \;=\; 1\,.$$

For any integer $k > i$,

$$f(2^k) - f(2^i) \;=\; \sum_{j=i}^{k-1} \left(f(2^{j+1} - f(2^j)\right)\,,$$

and hence,

$$\left|f(2^k) - f(2^i)\right| \;\leq\; \sum_{j=i}^{k-1} \left|f(2^{j+1}) - f(2^j)\right| \;\leq\; \sum_{j=i}^{k-1} 1 \;=\; k - i\,.$$

Consequently,

$$\sum_{i=1}^{k} \left|f(2^k) - f(2^i)\right| \;=\; \sum_{i=1}^{k-1} \left|f(2^k) - f(2^i)\right| \;\leq\; \sum_{i=1}^{k-1} (k-i) \;=\; \frac{k(k-1)}{2}\,.$$

3. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2 + 1$ for some integer m.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley's solution.

We will prove that the positive integers n which satisfy the given conditions are those of the form 2^t, for positive integers t. The following lemmas will be needed. For proofs of Lemmas 2 and 3, we refer to Niven, Zuckerman and Montgomery, *An Introduction to the Theory of Numbers* (Wiley, 1991).

Lemma 1. For any positive integer n, the positive integer $2^n - 1$ is divisible by 3 if and only if n is even.

Proof. If $n = 2m$ for some positive integer m, then $2^n = 2^{2m} = 4^m$. Thus, $2^n - 1 = 4^m - 1$ which is divisible by $4 - 1 = 3$.

If $n = 2m - 1$ for some positive integer m, then

$$2^n - 1 \;=\; 2^{2m-1} - 1 \;=\; 1 + 2 + 2^2 + \cdots + 2^{2m-2} \;\equiv\; 1 \pmod 3\,,$$

since $2^{2k-1} + 2^{2k} = 3(2^{2k-1}) \equiv 0 \pmod 3$ for all positive integers k.

Lemma 2. If q is a prime of the form $4k + 3$ and $q \mid a^2 + b^2$, then $q \mid a$ and $q \mid b$.

Lemma 3. If $N = a^2 + b^2$ is odd and contains no prime factors of the form $4k + 3$, then there exist *coprime* integers x, y such that $N = x^2 + y^2$. (In fact, the number of such representations is 2^{t+2}, where t is the number of primes p of the form $4k + 1$ that divide N.)

In view of Lemma 1, our problem reduces to finding all *even* positive integers n such that $\frac{1}{3}(2^n - 1)$ divides $4m^2 + 1$ for some integer m. Let $n = 2k$, where k is a positive integer. Then $\frac{1}{3}(2^n - 1) = \frac{1}{3}(4^k - 1)$. We will show that $\frac{1}{3}(4^k - 1)$ divides $4m^2 + 1$ for some integer m if and only if $k = 2^t$ for some integer t.

Let k be a positive integer, and let $N = \frac{1}{3}(4^k - 1)$.

First we suppose that $k = 2^t$ for some (non-negative) integer t. Then

$$\begin{aligned} N &= \tfrac{1}{3}(4^{2^t} - 1) \;=\; 1 + 4 + 4^2 + \cdots + 4^{(2^t-1)} \\ &= (1+4)(1+4^2)(1+4^4)\cdots(1+4^{2^{t-1}})\,, \end{aligned}$$

since $1 + 2 + 4 + \cdots + 2^{t-1} = 2^t - 1$. In this factorization of N, each factor $1 + 4^u = 1^2 + (2^u)^2$ is odd and is a sum of two squares. Therefore, N is odd and, by a well-known theorem, is expressible as a sum of two squares, one even and one odd. By Lemma 2, the factors $1 + 4^u$ contain no prime factor of the form $4k + 3$, since such a factor cannot divide 1. Therefore, N contains no prime factor of the form $4k + 3$. By Lemma 3, it follows that $N = x^2 + y^2$ for some x, y such that $(x, y) = 1$. Furthermore, one of x, y is odd and the other is even.

Since $(x, y) = 1$, there exist integers λ and μ such that $\lambda x + \mu y = 1$. For any such λ and μ,

$$\begin{aligned} N(\lambda^2 + \mu^2) &= (x^2 + y^2)(\lambda^2 + \mu^2) \\ &= (\lambda x + \mu y)^2 + (\lambda y - \mu x)^2 \\ &= 1 + (\lambda y - \mu x)^2\,. \end{aligned}$$

We will show that λ and μ may be chosen so that $\lambda y - \mu x$ is even. Then we will have $N(\lambda^2 + \mu^2) = 1 + 4m^2$ for an integer m, showing that N is a divisor of $1 + 4m^2$.

If λ_0 and μ_0 are particular values of λ and μ such that $\lambda_0 x + \mu_0 y = 1$, then the general λ and μ are given by $\lambda = \lambda_0 - ky$ and $\mu = \mu_0 + kx$, where k is any integer. Then $\lambda y - \mu x = \lambda_0 y - \mu_0 x - k(x^2 + y^2)$. Therefore, if $\lambda_0 y - \mu_0 x$ is odd, we can choose $k = 1$, making $\lambda y - \mu x$ even.

Now suppose k is not a power of 2. Then $k = 2^s t$, where t is odd and $t \geq 3$, and we have

$$N \;=\; \tfrac{1}{3}(4^{2^s t} - 1) \;=\; \tfrac{1}{3}(2^{2^{s+1}t} + 1)(2^{2^{s+1}t} - 1)\,.$$

The second factor is divisible by $2^t - 1$, which is congruent to $3 \pmod 4$ and to $1 \pmod 3$. In other words, this factor contains a prime factor p other than 3 such that $p \equiv 3 \pmod 4$. Hence, N contains this factor p. If N has the form $x^2 + y^2$, then x and y both have the factor p, by Lemma 2. Therefore, there is no multiple of N that is equal to $1 + 4m^2$ for some integer m (since p cannot divide 1).

4. Suppose that for any real x ($|x| \neq 1$), a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) \;=\; x\,.$$

Find all possible $f(x)$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution of Chen and Wang.

Direct computation shows that the given condition is satisfied by

$$f(x) \;=\; \frac{7x + x^3}{2(1-x^2)}\,, \qquad x \;\neq\; \pm 1\,.$$

We will prove that this is the only possible $f(x)$.

Let $f(x)$ be any function that satisfies the given condition. Let $x \in \mathbb{R}$ be arbitrary such that $x \neq \pm 1$. Set $y = \dfrac{x+3}{1-x}$. Then it is readily seen that $y \neq \pm 1$ and $x = \dfrac{y-3}{y+1}$. Furthermore, we find that $\dfrac{3+y}{1-y} = \dfrac{x-3}{x+1}$. Hence,

$$f(x) + f\left(\frac{x-3}{x+1}\right) \;=\; f\left(\frac{y-3}{y+1}\right) + f\left(\frac{3+y}{1-y}\right) \;=\; y\,;$$

that is,

$$f(x) + f\left(\frac{x-3}{x+1}\right) \;=\; \frac{x+3}{1-x}\,. \tag{1}$$

Now set $y = \frac{x-3}{x+1}$. It is readily seen that $y \neq \pm 1$ and $x = \frac{3+y}{1-y}$. Furthermore, $\frac{y-3}{y+1} = \frac{3+x}{1-x}$. Hence,

$$f\left(\frac{3+x}{1-x}\right) + f(x) \;=\; f\left(\frac{y-3}{y+1}\right) + f\left(\frac{3+y}{1-y}\right) \;=\; y\,;$$

that is,

$$f\left(\frac{3+x}{1-x}\right) + f(x) \;=\; \frac{x-3}{x+1}\,. \tag{2}$$

Adding (1) and (2), we then have

$$2f(x) + x \;=\; \frac{x+3}{1-x} + \frac{x-3}{x+1} \;=\; \frac{8x}{1-x^2}\,,$$

from which it follows that

$$f(x) \;=\; \frac{1}{2}\left(\frac{8x}{1-x^2} - x\right) \;=\; \frac{7x+x^3}{2(1-x^2)}\,.$$

5. Consider a permutation $a_1a_2a_3a_4a_5a_6$ of the 6 numbers $\{1, 2, 3, 4, 5, 6\}$ which can be transformed to $1\ 2\ 3\ 4\ 5\ 6$ by transposing two numbers exactly 4 times (not less than 4 times). Find the number of such permutations.

Solution by Christopher J. Bradley, Bristol, UK.

In the theory of permutations, every permutation may be expressed as a product of cycles. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 2 & 6 & 4 \end{pmatrix} \;=\; (1\ 5\ 6\ 4\ 2\ 3)\,,$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 1 & 3 & 2 \end{pmatrix} \;=\; (2\ 6)(1\ 5\ 3\ 4)\,.$$

There are $(n-1)!$ different n–cycles. For example, the number of different cycles of $\{1, 5, 3, 4\}$ is $3! = 6$, namely $(1\ 5\ 3\ 4)$, $(1\ 5\ 4\ 3)$, $(1\ 4\ 5\ 3)$, $(1\ 4\ 3\ 5)$, $(1\ 3\ 4\ 5)$, and $(1\ 3\ 5\ 4)$. (Any one of the numbers may be fixed in the first position.)

Furthermore, each permutation may be expressed as a product of transpositions (in many ways). For an n–cycle, this requires a minimum of $(n-1)$ transpositions. For example, $(1\ 5\ 3\ 4) = (1\ 4)(1\ 3)(1\ 5)$.

The permutation group on 6 symbols may be split into classes corresponding to the 11 partitions of the integer 6, as shown in the table below. The number of permutations in each class is indicated, along with the minimum number of transpositions required to express any element in the class as a product of transpositions. The total number of permutations for which the minimum number of transpositions is exactly four is $144+90+40 = 274$.

	Number of Transpositions	Number of Permutations		
one 6–cycle	5	$5!$	$=$	120
one 5–cycle, one 1–cycle	4	$6 \times 4!$	$=$	144
one 4–cycle, one 2–cycle	4	$15 \times 3!$	$=$	90
one 4–cycle, two 1–cycles	3	$15 \times 3!$	$=$	90
two 3–cycles	4	$10 \times 2! \times 2!$	$=$	40
one 3–cycle, two 2–cycle, one 1–cycle	3	$60 \times 2!$	$=$	120
one 3–cycle, three 1–cycles	2	$20 \times 2!$	$=$	40
three 2–cycles	3	15	$=$	15
two 2–cycles, two 1–cycles	2	45	$=$	45
one 2–cycle, four 1–cycles	1	15	$=$	15
six 1–cycles	0	1	$=$	1
		Total	$=$	720

[*Ed.* It is unclear in the problem statement whether we should be requiring the *minimum* number of transpositions to be equal to 4. Perhaps we should allow any transformation that can be obtained by exactly four transpositions, whether this is the minimum number or not. In this case, we should include those classes in the table for which the minimum number of transpositions is 0 or 2. (We can always apply the same transposition twice in succession with no net effect.) The answer is then $274 + 40 + 45 + 1 = 360$.]

To complete this number of the *Corner*, we turn to solutions from our readers to problems of the Grosman Memorial Mathematical Olympiad 1999 given [2001 : 423–424].

1. For every 16 positive integers n, a_1, a_2, $\ldots$, a_{15} we define

$$T(n, a_1, a_2, \ldots, a_{15}) \;=\; (a_1^n + a_2^n + \cdots + a_{15}^n) a_1 a_2 \ldots a_{15}\,.$$

Find the smallest n for which $T(n, a_1, a_2, \ldots, a_{15})$ is divisible by 15 for every choice of a_1, a_2, $\ldots$, a_{15}.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

We will show that $n = 4$ is the smallest n with the desired property.

First suppose that $n = 4$. Let a_1, a_2, $\ldots$, a_{15} be positive integers, and let $b = T(n, a_1, a_2, \ldots, a_{15})$. For each $i = 1, 2, \ldots, 15$, either $a_i \equiv 0 \pmod 3$ or $a_i^4 = a_i^2 \times a_i^2 \equiv 1 \pmod 3$, by Fermat's Little Theorem. If $a_i \equiv 0 \pmod 3$ for some i, then $b \equiv 0 \pmod 3$, since a_i is a factor of b. If $a_i \not\equiv 0 \pmod 3$ for all i, then $a_1^4 + a_2^4 + \cdots + a_{15}^2 \equiv 15 \equiv 0 \pmod 3$, and again we see that $b \equiv 0 \pmod 3$.

Similarly, for each $i = 1, 2, \ldots, 15$, either $a_i \equiv 0 \pmod 5$ or $a_i^4 \equiv 1 \pmod 5$, giving us $b \equiv 0 \pmod 5$. Therefore, $b \equiv 0 \pmod{15}$. Hence, $n = 4$ has the desired property.

Now consider positive integers $n < 4$. To see that $n = 1$ does not have the desired property, note that

$$T(1,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \;=\; 22 \times 128 \;\not\equiv\; 0 \pmod{15} \;.$$

Since $1^2 \equiv 1 \pmod 5$ and $2^2 \equiv -1 \pmod 5$, it is easy to verify that

$$T(2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \;\equiv\; 1 \times 2^7 \;\not\equiv\; 0 \pmod 5 \;.$$

Then $T(2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \not\equiv 0 \pmod{15}$. Thus, $n = 2$ does not have the desired property.

Since $1^3 \equiv 1 \pmod 3$ and $2^3 \equiv -1 \pmod 3$, it is easy to verify that

$$T(3,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \;\equiv\; 1 \times 2^7 \;\not\equiv\; 0 \pmod 3 \;.$$

Then $T(3,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \not\equiv 0 \pmod{15}$. Thus, $n = 3$ does not have the desired property.

2. Find the smallest integer n for which $0 \;<\; \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor \;<\; 10^{-5}$.

Remark. $\lfloor x \rfloor$ denotes the integral value of x; that is, the largest integer which does not exceed x.

Solution by Christopher J. Bradley, Bristol, UK.

Let $f(n) = n^{1/4} - \lfloor n^{1/4} \rfloor$, for positive integers n. If $n = x^4$ for some positive integer x, then $f(n) = x - x = 0$. Moreover, $f(n)$ is clearly increasing on $[x^4, (x+1)^4)$. Hence, in order to get the *smallest* integer n such that $0 < f(n) < 10^{-5}$, we must choose $n = x^4 + 1$, with x as small as possible.

Now,

$$f(x^4+1) \;=\; (1+x^4)^{1/4} - x \;=\; x\left(1 + \frac{1}{x^4}\right)^{1/4} - x \,,$$

which, by the Binomial Theorem, yields

$$\begin{aligned} f(x^4+1) \;&=\; x\left(1 + \frac{1}{4}\frac{1}{x^4} + \frac{1}{4}\left(\frac{-3}{4}\right)\frac{1}{2!}\frac{1}{x^8} + \cdots\right) - x \\ &=\; \frac{1}{4x^3} - \frac{3}{32}\frac{1}{x^7} + \cdots \end{aligned}$$

Therefore, we look for x such that $\frac{1}{4x^3} < 10^{-5}$; that is, $4x^3 > 10^5$. Since $4 \times 30^3 = 1.08 \times 10^5$, and a value of $x = 30$ makes the subsequent terms in the binomial expansion negligible compared with 10^{-5}, it follows that $x = 30$ and $n = 30^4 + 1 = 810001$.

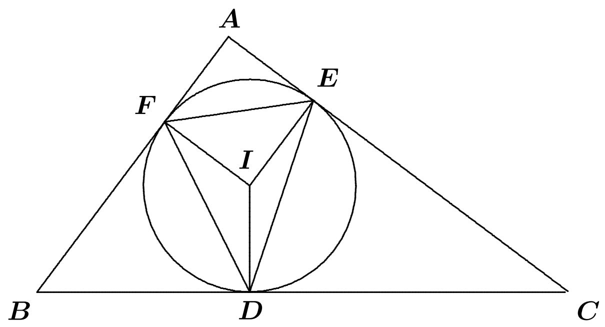
3. For every triangle ABC, denote by $D(ABC)$, the triangle whose vertices are the tangency points of the incircle of ABC (touching the sides of the triangle). The given triangle ABC is not equilateral.

(a) Prove that $D(ABC)$ is also not equilateral.

(b) Find in the sequence of triangles $T_1 = \triangle ABC$, $T_{k+1} = D(T_k)$, $k = 1, 2, \ldots$ a triangle whose largest angle α satisfies the inequality $0 < \alpha - 60^\circ < 0.0001$.

Solution by Christopher J. Bradley, Bristol, UK, modified by the editor.

(a) Let I be the incentre of $\triangle ABC$, and let D, E, F be the points of tangency of the incircle opposite A, B, C, respectively, as in the diagram below. Thus, $D(ABC) = \triangle DEF$.



Since $\angle IFA = \angle IEA = 90^\circ$, we have $\angle FIE = 180^\circ - A = B + C$ and $\angle FDE = \frac{1}{2}\angle FIE = \frac{1}{2}(B + C)$. Similarly, $\angle DEF = \frac{1}{2}(A + C)$ and $\angle EFD = \frac{1}{2}(A + B)$.

If $\triangle DEF$ is equilateral, then $A + B = A + C = B + C = 120^\circ$, from which it follows that $A = B = C = 60^\circ$, and $\triangle ABC$ is equilateral. Hence, since $\triangle ABC$ is not equilateral, neither is $\triangle DEF$.

(b) For each $k = 1, 2, \ldots$, let the vertices of T_k be A_k, B_k, C_k, and let $\alpha_k = \angle A_k$, $\beta_k = \angle B_k$, and $\gamma_k = \angle C_k$. We may suppose, without loss of generality, that the vertices are labelled so that $\alpha_k \geq \beta_k \geq \gamma_k$. Then, from the proof of part (a), we see that

$$\alpha_{k+1} = \tfrac{1}{2}(\alpha_k + \beta_k), \qquad \beta_{k+1} = \tfrac{1}{2}(\alpha_k + \gamma_k), \qquad \gamma_{k+1} = \tfrac{1}{2}(\beta_k + \gamma_k),$$

for $k = 1, 2, \ldots$. By solving this system of difference equations, we obtain for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\alpha_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\alpha_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\beta_1 + \gamma_1), \\
\beta_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\beta_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\alpha_1 + \gamma_1), \\
\gamma_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\gamma_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\alpha_1 + \beta_1).
\end{aligned}$$

We can then simplify the largest angle:

$$\begin{aligned}\alpha_{2n+1} &= \frac{1}{3}\left(1+\frac{1}{2^{2n-1}}\right)\alpha_1 + \frac{1}{3}\left(1-\frac{1}{2^{2n}}\right)(180^\circ - \alpha_1) \\ &= \left(1-\frac{1}{2^{2n}}\right)60^\circ + \left(\frac{1}{2^{2n}}\right)\alpha_1 \\ &= 60^\circ + \frac{1}{2^{2n}}(\alpha_1 - 60^\circ)\,.\end{aligned}$$

Since $60^\circ < \alpha_1 < 180^\circ$, we have $60^\circ < \alpha_{2n+1} < 60^\circ + \frac{1}{2^{2n}}120^\circ$. To obtain $0 < \alpha_{2n+1} - 60^\circ < 0.0001$, it is sufficient to choose n so that $120/2^{2n} < 0.0001$; that is, $2^{2n} > 1200000$. The smallest such n is 11, since $2^{20} \approx 1024^2 \approx 1000000$. Hence, the largest angle of triangle T_{23} satisfies the given inequality.

4. Consider a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients a, b, c, d. Prove that if $f(x)$ has exactly one real root then $f(x)$ can be factored into terms with rational coefficients.

Solution by Christopher J. Bradley, Bristol, UK..

Since the coefficients are real, complex roots occur in conjugate pairs. Hence, if there is only one real root it must be either of multiplicity 2 or of multiplicity 4. If it is of multiplicity 4, then $f(x)$ must factor as $(x-k)^4$, where $a = 4k$, $b = 6k^2$, $c = 4k^3$, and $d = k^4$; thus, $k = a/4$ is rational (and must be an integer, since k^4 is an integer).

If x_0 is a real root of multiplicity 2, then $f(x_0) = 0$ and $f'(x_0) = 0$, and we have

$$x_0^4 + ax_0^3 + bx_0^2 + cx_0 + d \;=\; 0\,, \tag{1}$$

$$4x_0^3 + 3ax_0^2 + 2bx_0 + c \;=\; 0\,. \tag{2}$$

Subtracting x_0 times equation (2) from 4 times equation (1) gives

$$ax_0^3 + 2bx_0^2 + 3cx_0 + 4d \;=\; 0\,. \tag{3}$$

Subtracting 4 times equation (3) from a times equation (2) gives

$$(3a^2 - 8b)x_0^2 + 2(ab - 6c)x_0 + (ac - 16d) \;=\; 0\,. \tag{4}$$

Subtracting $4x_0$ times equation (4) from $(3a^2 - 8b)$ times equation (2) gives

$$(9a^3 - 32ab + 48c)x_0^2 + 2(3ba^2 - 8b^2 - 2ac + 32d)x_0\,, c(3a^2 - 8b) \;=\; 0\,. \tag{5}$$

Now x_0^2 can be eliminated from (4) and (5) to deduce the value of x_0, which we perceive to be rational.

We now quote the result, which is well known, that if a, b, c, d are integers, then any rational root of the polynomial $x^4 + ax^3 + bx^2 + cx + d = 0$ is integral. We conclude that x_0 is an integer.

Now

$$x^4 + ax^3 + bx^2 + cx + d \;=\; (x - x_0)^2 q(x)\,, \qquad (6)$$

where $q(x)$ is quadratic. This quadratic is irreducible over the reals, since the roots of $q(x) = 0$ are complex. Furthermore, from (6), since x_0 is integral, $q(x)$ is of the form $x^2 + Ax + B$, where A and B are rational. In fact, since $A - 2x_0 = a$ and $B = 2x_0A + b - x_0^2$, we see that A and B are integers.

[*Ed.* The reduction process that produced equations (3) to (5) is essentially the Euclidean Algorithm being used to find the greatest common divisor of the polynomials $f(x)$ and $f'(x)$. Since both $f(x)$ and $f'(x)$ have rational coefficients, each new polynomial generated by the algorithm also has rational coefficients. Thus, even though we may look for the greatest common divisor in $\mathbb{R}[x]$, we end up with an element of $\mathbb{Q}[x]$. We know that the greatest common divisor of $f(x)$ and $f'(x)$ in $\mathbb{R}[x]$ is $x - x_0$. We conclude that $x - x_0 \in \mathbb{Q}[x]$; that is, x_0 is rational. This argument can be applied more generally to show that whenever a polynomial $f(x) \in \mathbb{Q}[x]$ has a single repeated root in any extension field of $\mathbb{Q}$ (for example, $\mathbb{R}$ or $\mathbb{C}$), that root must in fact be rational.]

5. An infinite sequence of distinct real numbers is given. Prove that it contains a subsequence of 1999 terms which is either monotonically increasing or monotonically decreasing.

Remark. The sequence of numbers $a_1, \ldots, a_n$ is said to be monotone increasing if $a_1 < a_2 < \cdots < a_n$ and monotone decreasing if $a_1 > a_2 > \cdots > a_n$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

Let $\{a(n)\}$ be the given sequence and let

$$S \;=\; \{m \in \mathbb{N} \mid a(m) > a(n) \text{ for all } n > m\}\,.$$

Two mutually exclusive cases can occur:

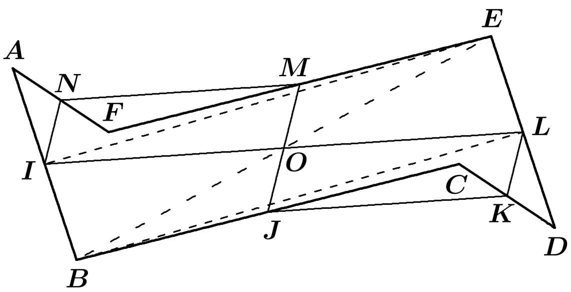
Case (i). S is infinite. Then we can certainly choose $m_1, m_2, \ldots, m_{1999}$ in S such that $m_1 < m_2 < \cdots < m_{1999}$. By definition of S, the subsequence $\{a(m_j)\}_{1 \le j \le 1999}$ is decreasing.

Case (ii). S is finite. Then there exists $N_1 \in \mathbb{N}$ such that $N_1 \not\in S$. Since $N_1 \not\in S$, there exists $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and $a(N_1) \le a(N_2)$; we even have $a(N_1) < a(N_2)$, since the terms of the sequence are distinct. Iterating the process, we determine $N_1 < N_2 < \cdots < N_{1999}$ such that $a(N_1) < a(N_2) < \cdots < a(N_{1999})$. The subsequence $\{a(N_j)\}_{1 \le j \le 1999}$ is increasing.

6. Six points A, B, C, D, E, F are given in space. The quadrilaterals $ABDE$, $BCEF$, $CDFA$ are parallelograms. Prove that the six mid-points of the sides AB, BC, CD, DE, EF, FA are coplanar.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We use Bataille's solution.

Let I, J, K, L, M, N be the mid-points of AB, BC, CD, DE, EF, FA, respectively, and let $\mathcal{P}$ and $\mathcal{Q}$ be the planes determined by M, N, I and J, K, L, respectively. We are required to prove that $\mathcal{P} = \mathcal{Q}$. It suffices to show that the lines IL and JM are both contained in $\mathcal{P}$ and $\mathcal{Q}$ and are concurrent.



Let O be the mid-point of BE. Then, $IOMN$ is the Varignon Parallelogram of $ABEF$; hence, $O \in \mathcal{P}$. Similarly, $O \in \mathcal{Q}$ [using $JKLO$]. Since $\overrightarrow{IB} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{ED} = \overrightarrow{EL}$, we see that $IBLE$ is a parallelogram, and therefore O is the mid-point of IL. Thus, the line IL is contained in $\mathcal{P}$ (since O and I are in $\mathcal{P}$) and in $\mathcal{Q}$ (since O and L are in $\mathcal{Q}$). In a similar way, we see that O is the mid-point of JM. It follows that JM is contained in both $\mathcal{P}$ and $\mathcal{Q}$. To complete the proof, we observe that IL and JM clearly concur at O.

That completes the Olympiad Corner for this issue. Send me your contests as well as your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

The Countingbury Tales: Fun with Mathematics
By Miguel de Guzmán, translated by Jody Doran, published by World Scientific, 2000
ISBN 981-02-4033-3, softcover, 121 pages, US$21.00.
Reviewed by ***Sarah McCurdy****, student in the Faculty of Education, University of New Brunswick, Fredericton, NB.*

This entertaining little book is composed of nine short chapters, each of which presents a mathematical puzzle or phenomenon. Most of the "tales" are well known, such as an examination of the four colour map problem or the game of solitaire. The sections are generally accessible to a motivated or guided high school student and any mathematics undergraduate. The book would be a valuable resource for a high school mathematics teacher looking for enrichment material or a source of projects. The chapters often guide the reader through a problem, encouraging him or her to pause and work through the details of an argument. The tone is friendly and playful as the reader is encouraged step by step, but this easy manner can lead to a casual attitude towards definitions of new terms and ideas.

Miguel de Guzmán starts his book by commenting that many profound ideas in mathematics are born out of curious games and puzzles which are passed from one person to another. Mathematicians play around with ideas and observe orderly patterns, and this can often lead to new branches of thought. Citing the Königsberg bridge problem, as well as Pascal's and Fermat's investigations of probability as examples, de Guzmán states "games and beauty are found in the origin of a major part of mathematics." Thus, he proposes to teach some aspects of mathematics though similar games and observations. This promising idea leads to four sections on games (Nim, Solitaire, Leap Frog, and a domino-laying game), two on graph theory topics (the Four-Colour Problem and the Königsberg bridges), and, somewhat surprisingly, two that demonstrate some geometrical oddities of ellipses and parabolas. The remaining chapter, "The Mathematician as a Naturalist," details the observations that can be made to deduce the five Platonic solids. In all these sections, curiosity is the prevailing attitude, and de Guzmán uncovers surprising facts with verve.

What makes these tales shine is the inclusion of historical background. The majority of the chapters describe a puzzle and the historical setting in which it was introduced or investigated. The characters are briefly sketched as well-rounded people, not just as mathematicians. These people are portrayed as curious problem-solvers who collaborate with others. Their hard work and dedication are mentioned, emphasizing that mathematics is a process which rewards persistence.

The chapters are diverse in level. Some are above high school level, such as the chapters on conics sections; one of these has, in its opening paragraph, the sentence, "Remember: an ellipse is a set in the plane made up of all the points the sum of whose distances to two fixed points, the foci, is constant." This is a fact which many high school students and even some first-year university students do not have at hand. Other chapters need virtually no outside information and serve as the pleasant introductions they were intended to be. The chapter on Nim, however, lacks a proper explanation of binary notation and the nimsum that is essential to the analysis of the game.

Generally, the chapters contain work for readers to do themselves. Even more valuable are the hints for further work, which would usually be a good jumping-off point for independent study or a project. In fact, as a source of project ideas, the book is ideal. The topics are not those generally studied in school, and the book usually provides enough background and a "hook" to get students started. Unfortunately, some of the vocabulary is non-standard, which might cause students to hit a few snags as they proceed from the chapter to their research. As well, the book is desperately in need of a glossary to accompany the new terms which are used but not well explained or defined. These two problems could be addressed by the teacher acting as a resource, which would also clear up the few places where the translation is awkward.

Overall, this is a useful book that enlivens some branches of math not normally addressed in schools. The book is accessible and interesting in most places. It can move quickly at times, but the reader is rewarded when he or she works through the problems and ideas presented.

The Contest Problem Book VI: American High School Mathematics Examination (AHSME) 1989–1994
Compiled and augmented by Leo J. Schneider, published by the Mathematics Association of America, 2000
ISBN 0-88385-642-5, paperbound, 212 pages, US$25.95.
Reviewed by **John Grant McLoughlin**, *Faculty of Education, University of New Brunswick, Fredericton, NB.*

The book features a comprehensive set of problems and detailed solutions for each of AHSME contests 40 through 45 inclusive. Select comments on the distractors along with detailed answer and response distributions are included with the various sets of problems. The book concludes with three additional features/chapters, namely, an insider's look at the problems, some helpful tools for problem solving, and finally, a classification of the problems by topic. The book is well suited to those seeking problems to use with contest preparation or independent problem solving development of keen high school students.

Some Necessary Conditions for a Real Polynomial to have only Real Roots

C.H. Harris Kwong and Amitabha Tripathi

The purpose of this note is to derive some necessary conditions for a real polynomial of degree greater than one to have all its roots real. While there are several articles [2, 3, 4, 5] that deal with related problems, our results are simple and elementary.

Theorem 1 Let $n \geq 2$, and let a_0, a_1, $\ldots$, a_n be real numbers with $a_n \neq 0$. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-1)a_1^2 \geq 2na_0a_2$.

Proof. If $a_0 = 0$, we have nothing to prove. Hence, we may assume $a_0 \neq 0$, so that all the roots of $p(x)$ are non-zero. Since the roots of $p(x)$ are real, so are the roots of

$$q(x) \;=\; x^n p(1/x) \;=\; a_n + a_{n-1}x + \cdots + a_2x^{n-2} + a_1x^{n-1} + a_0x^n\,.$$

It follows from Rolle's Theorem that all the roots of $q^{(k)}(x)$ are real for each k, where $0 \leq k \leq n-1$. In particular, the roots of

$$q^{(n-2)}(x) \;=\; (n-2)!\,a_2 + (n-1)!\,a_1x + \frac{n!}{2}\,a_0x^2$$

are real. This implies

$$[(n-1)!\,a_1]^2 \;\geq\; 4\cdot(n-2)!\,a_2\cdot\frac{n!}{2}\,a_0\,,$$

or equivalently, $(n-1)a_1^2 \geq 2na_0a_2$. □

The inequality in Theorem 1 is also a sufficient condition for all roots to be real if $n = 2$. However, this condition is no longer sufficient when $n \geq 3$. For instance,

$$p(x) \;=\; x(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

has roots 0 and $e^{\frac{2\pi ik}{n}}$, where $1 \leq k \leq n-1$, although $(n-1)a_1^2 \geq 2na_0a_2$.

We can extend Theorem 1 to any three consecutive coefficients.

Theorem 2 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-k-1)(k+1)a_{k+1}^2 \geq (n-k)(k+2)a_k a_{k+2}$ for each k, where $0 \leq k \leq n-2$.

Proof. Since the roots of

$$p^{(k)}(x) = \sum_{m=k}^{n} \binom{m}{k} k!\, a_m x^{m-k}$$

are real, Theorem 1 implies that

$$(n-k-1)\left[\binom{k+1}{k} k!\, a_{k+1}\right]^2 \geq 2(n-k)\left[\binom{k}{k} k!\, a_k\right]\left[\binom{k+2}{k} k!\, a_{k+2}\right],$$

so that

$$(n-k-1)(k+1)a_{k+1}^2 \geq (n-k)(k+2)a_k a_{k+2}\,. \tag{1}$$

This completes the proof. □

Corollary 1 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be positive real numbers. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $a_1 a_{n-1} \geq n^2 a_0 a_n$.

Proof. From inequality (1), we have

$$\prod_{k=0}^{n-2} \frac{k+1}{k+2} \prod_{k=0}^{n-2} \frac{a_{k+1}}{a_k} \geq \prod_{k=0}^{n-2} \frac{n-k}{n-k-1} \prod_{k=0}^{n-2} \frac{a_{k+2}}{a_{k+1}},$$

which reduces to

$$\frac{1}{n} \cdot \frac{a_{n-1}}{a_0} \geq \frac{n}{1} \cdot \frac{a_n}{a_1}\,.$$

Hence, $a_1 a_{n-1} \geq n^2 a_0 a_n$. □

Next we have an interesting proof of a slightly weaker version of Theorem 1.

Theorem 3 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $na_1^2 \geq 8a_0 a_2$.

Proof. We may assume $a_0 a_2 \neq 0$, for otherwise there is nothing to prove. Thus, all roots of $p(x)$ are real and non-zero; let them be $\alpha_1, \alpha_2, \ldots, \alpha_n$. For each j, set $m_j = a_n \alpha_j^{n-2} + a_{n-1}\alpha_j^{n-3} + \cdots + a_3\alpha_j + a_2$. Then we have

$$0 \;=\; p(\alpha_j) \;=\; m_j \alpha_j^2 + a_1 \alpha_j + a_0 \,. \tag{2}$$

Thus, the real quadratic polynomial $m_j x^2 + a_1 x + a_0$ has a real root α_j. Therefore,

$$a_1^2 \;\geq\; 4a_0 m_j \qquad \text{for} \quad 1 \leq j \leq n\,. \tag{3}$$

Summing inequality (3) over all j, we have, because of equation (2),

$$n a_1^2 \;\geq\; 4a_0 \sum_{j=1}^{n} m_j \;=\; -4a_0 a_1 \sum_{j=1}^{n} \alpha_j^{-1} - 4a_0^2 \sum_{j=1}^{n} \alpha_j^{-2}\,. \tag{4}$$

Since α_j^{-1} are the roots of the polynomial $q(x) = x^n p(1/x) = \sum_{i=0}^{n} a_i x^{n-i}$, we have

$$\sum_{j=1}^{n} \alpha_j^{-1} \;=\; -\frac{a_1}{a_0}\,. \tag{5}$$

From $\sum_{j=1}^{n} \alpha_j^{-2} = \Big(\sum_{j=1}^{n} \alpha_j^{-1}\Big)^2 - 2 \sum_{1\leq j<k\leq n} (\alpha_j \alpha_k)^{-1}$, we find

$$\sum_{j=1}^{n} \alpha_j^{-2} \;=\; \Big(-\frac{a_1}{a_0}\Big)^2 - 2\,\frac{a_2}{a_0}\,, \tag{6}$$

The proof is completed by substituting (5) and (6) into (4). □

We derive from Theorem 3, in a manner analogous to Theorem 2 and Corollary 1, the following results.

Theorem 4 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-k)(k+1)a_{k+1}^2 \;\geq\; 4(k+2)a_k a_{k+2}$ for each k, where $0 \leq k \leq n-2$.

Corollary 2 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be positive real numbers. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-1)!\, a_1 a_{n-1} \geq 4^{n-1} a_0 a_n$.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er octobre 2004**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Hidemitsu Saeki, de l'Université de Montréal, d'avoir traduit les problèmes.

2914. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Sur les côtés d'un triangle acutangle ABC, on construit extérieurement des triangles isocèles de même type, DBC, ECA et FAB, de sorte que

$$\begin{aligned} \angle DBC &= \angle DCB = \angle EAC = \angle ECA \\ &= \angle FAB = \angle FBA = \angle BAC\,. \end{aligned}$$

Soit M le point milieu de BC, P et Q les intersections respectives de DE avec AC et de DF avec AB.

Montrer que $MP : MQ = AB : AC$.

2915. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On donne un triangle ABC avec $AB < AC$ et soit I le centre du cercle inscrit, M le point milieu de BC. Supposons que D soit l'intersection de IM avec AB et que E soit l'intersection avec CI de la perpendiculaire abaissée de B sur AI .

Montrer que $DE \parallel AC$.

2916. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit $S = A_1A_2A_3A_4$ un tétrahèdre et M le point de Steiner, c'est-à-dire le point tel que $\sum\limits_{j=1}^{4} A_jM$ soit minimal. Si M est un point intérieur de S et si A_j' dénote l' intersection de A_jM avec la face opposée, montrer que

$$\sum_{j=1}^{4} A_jM \ \geq\ 3\sum_{j=1}^{4} A_j'M\,.$$

2917★. *Proposé par Šefket Arslanagić et Faruk Zejnulahi, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Si x_1, x_2, x_3, x_4, $x_5 \geq 0$ et $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, montrer ou réfuter l'inégalité

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \frac{x_3}{1+x_4} + \frac{x_4}{1+x_5} + \frac{x_5}{1+x_1} \geq \frac{5}{6}\,.$$

2918. *Proposé par Šefket Arslanagić et Faruk Zejnulahi, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit a_1, a_2, $\ldots$, a_{100} des nombres réels satisfaisant :

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} &\geq 0\,; \\ a_1^2 + a_2^2 &\geq 200\,; \\ a_3^2 + a_4^2 + \cdots + a_{100}^2 &\geq 200\,. \end{aligned}$$

Quelle est la valeur minimale de $a_1 + a_2 + \cdots + a_{100}$?

2919★. *Proposé par Ross Cressman, Université Wilfrid Laurier, Waterloo, ON.*

Soit $n \in \mathbb{N}$ avec $n > 1$, et soit

$$T_n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_j > 0 \text{ for } j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} x_j = 1 \right\}.$$

Soit p, q et $r \in T_n$ tels que $\sum\limits_{j=1}^{n} \sqrt{q_j r_j} < \sum\limits_{j=1}^{n} \sqrt{p_j r_j}$.

Confirmer ou infirmer que :

(a) $\sum\limits_{j=1}^{n} \sqrt{q_j\,(r_j + p_j)} \;<\; \sum\limits_{j=1}^{n} \sqrt{p_j\,(r_j + p_j)}$,

(b) pour tout $\lambda \in [0,1]$,

$$\sum_{j=1}^{n} \sqrt{q_j\,(\lambda r_j + (1-\lambda) p_j)} \;<\; \sum_{j=1}^{n} \sqrt{p_j\,(\lambda r_j + (1-\lambda) p_j)}\,.$$

[Remarques du proposeur : (a) est un cas spécial de (b) avec $\lambda = \frac{1}{2}$. Cette question est reliée aux propriétés de la métrique de Shahshahani sur T_n, une métrique importante pour la génétique de populations.]

2920. *Proposé par Simon Marshall, étudiant, Onslow College, Wellington, Nouvelle-Zélande.*

Soit a, b et c des nombres réels positifs. Montrer que

$$a^4 + b^4 + c^4 + 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) \ \geq\ 3\left(a^3b + b^3c + c^3a\right) .$$

2921. *Proposé par Barry R. Monson, Université de Nouveau-Brunswick, Fredericton, NB ; et J. Chris Fisher, Université de Regina, Regina, SK.*

A l'aide de *Cinderella*™ et *Lénárt sphere*™ on peut de nos jours faire d'authentiques constructions sphériques, utilisant une règle sphérique pour dessiner le grand cercle passant par deux points A and B, et des compas sphériques pour dessiner le cercle de centre A et de rayon BC ($\leq \frac{\pi}{2}$, par exemple, sur une sphère unité).

Donner une construction sphérique simple pour les sommets d'un icosahèdre régulier inscrit dans une sphère.

2922. *Proposé par Michel Bataille, Rouen, France.*

Si n est un entier non négatif, trouver une formule fermée pour la somme

$$\sum_{k=0}^{n}(-1)^k 2^k \binom{n}{k}\binom{2n-k}{n} .$$

2923. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Supposons que x, $y \geq 0$ (x, $y \in \mathbb{R}$) et $x^2 + y^3 \geq x^3 + y^4$. Montrer que $x^3 + y^3 \leq 2$.

2924. *Proposé par Todor Mitev, Université de Rousse, Rousse, Bulgarie.*

Soit x_1, …, x_n ($n \geq 3$) des nombres réels positifs satisfaisant

$$\frac{1}{1 + x_2^2 x_3 \cdots x_n} + \frac{1}{1 + x_1 x_3^2 \cdots x_n} + \cdots + \frac{1}{1 + x_1^2 x_2 \cdots x_{n-1}} \ \geq\ \alpha\,, \quad (1)$$

pour un certain $\alpha > 0$. Montrer que

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \ \geq\ \frac{n\alpha}{n-\alpha} x_1 x_2 \cdots x_n \,. \quad (2)$$

2925. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier tel que $n \geq 3$. Trouver les zéros de la fonction

$$f_n(x) \;=\; \sum_{k=1}^{n-1} \frac{\sin(k\pi/n)}{\sin\bigl((k\pi/n)-x\bigr)}\,.$$

....

2914. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

On the sides of an acute-angled triangle ABC, similar isosceles triangles DBC, ECA, FAB are constructed externally, such that

$$\begin{aligned}\angle DBC \;&=\; \angle DCB \;=\; \angle EAC \;=\; \angle ECA \\ &=\; \angle FAB \;=\; \angle FBA \;=\; \angle BAC\,.\end{aligned}$$

Let M be the mid-point of BC, and let P and Q be the intersections of DE with AC and of DF with AB, respectively.

Prove that $MP : MQ \;=\; AB : AC$.

2915. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB < AC$, let I be its incentre and let M be the mid-point of BC. Suppose that D is the intersection of IM with AB and that E is the intersection of CI with the perpendicular from B to AI.

Prove that $DE \parallel AC$.

2916. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S = A_1A_2A_3A_4$ be a tetrahedron and let M be the Steiner point; that is, the point M is such that $\sum\limits_{j=1}^{4} A_jM$ is minimized. Assuming that M is an interior point of S, and denoting by A_j' the intersection of A_jM with the opposite face, prove that

$$\sum_{j=1}^{4} A_jM \;\geq\; 3\sum_{j=1}^{4} A_j'M\,.$$

2917★. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

If x_1, x_2, x_3, x_4, $x_5 \geq 0$ and $x_1+x_2+x_3+x_4+x_5 = 1$, prove or disprove that

$$\frac{x_1}{1+x_2}+\frac{x_2}{1+x_3}+\frac{x_3}{1+x_4}+\frac{x_4}{1+x_5}+\frac{x_5}{1+x_1} \;\geq\; \frac{5}{6}\,.$$

2918. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a_1, a_2, $\ldots$, a_{100} be real numbers satisfying:

$$\begin{aligned} a_1 \;\geq\; a_2 \;\geq\; \cdots \;\geq\; a_{100} &\;\geq\; 0\,; \\ a_1^2 + a_2^2 &\;\geq\; 200\,; \\ a_3^2 + a_4^2 + \cdots + a_{100}^2 &\;\geq\; 200\,. \end{aligned}$$

What is the minimum value of $a_1 + a_2 + \cdots + a_{100}$?

2919★. *Proposed by Ross Cressman, Wilfrid Laurier University, Waterloo, ON.*

Let $n \in \mathbb{N}$ with $n > 1$, and let

$$T_n \;=\; \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_j > 0 \text{ for } j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} x_j = 1 \right\}.$$

Let p, q, $r \in T_n$ such that $\sum\limits_{j=1}^{n} \sqrt{q_j r_j} < \sum\limits_{j=1}^{n} \sqrt{p_j r_j}$.

Prove or disprove:

(a) $\sum\limits_{j=1}^{n} \sqrt{q_j\,(r_j + p_j)} \;<\; \sum\limits_{j=1}^{n} \sqrt{p_j\,(r_j + p_j)}$,

(b) for all $\lambda \in [0, 1]$,

$$\sum_{j=1}^{n} \sqrt{q_j\,(\lambda r_j + (1-\lambda) p_j)} \;<\; \sum_{j=1}^{n} \sqrt{p_j\,(\lambda r_j + (1-\lambda) p_j)}\,.$$

[Proposer's remarks: (a) is the special case of (b) with $\lambda = \frac{1}{2}$. This question is connected with properties of the Shahshahani metric on T_n, a metric important for population genetics.]

2920. *Proposed by Simon Marshall, student, Onslow College, Wellington, New Zealand.*

Let a, b, and c be positive real numbers. Prove that

$$a^4 + b^4 + c^4 + 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) \;\geq\; 3\left(a^3b + b^3c + c^3a\right)\,.$$

2921. *Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB; and J. Chris Fisher, University of Regina, Regina, SK.*

These days, with *Cinderella*™ and the *Lénárt sphere*™ at hand, one can do actual spherical constructions, using a spherical ruler to draw the complete great circle through points A and B, and spherical compasses to draw the circle with centre A and radius BC ($\leq \frac{\pi}{2}$, say, on a unit sphere).

Give a simple spherical construction for the vertices of a regular icosahedron inscribed in the sphere.

2922. *Proposed by Michel Bataille, Rouen, France.*

Suppose that n is a non-negative integer. Find a closed expression for

$$\sum_{k=0}^{n}(-1)^k 2^k \binom{n}{k}\binom{2n-k}{n}.$$

2923. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that $x, y \geq 0$ ($x, y \in \mathbb{R}$) and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

2924. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n$ ($n \geq 3$) are positive real numbers satisfying

$$\frac{1}{1+x_2^2x_3\cdots x_n} + \frac{1}{1+x_1x_3^2\cdots x_n} + \cdots + \frac{1}{1+x_1^2x_2\cdots x_{n-1}} \;\geq\; \alpha\,, \quad (1)$$

for some $\alpha > 0$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \;\geq\; \frac{n\alpha}{n-\alpha}x_1x_2\cdots x_n\,. \quad (2)$$

2925. *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 3$. Determine the zeros of the function

$$f_n(x) \;=\; \sum_{k=1}^{n-1}\frac{\sin(k\pi/n)}{\sin\big((k\pi/n)-x\big)}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2814. [2003 : 110] *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\,.$$

I. *Solution by D. Kipp Johnson, Beaverton, OR, USA.*

If a, b, c are positive real numbers with $a+b+c=abc$, then we may write $a=\tan x$, $b=\tan y$, $c=\tan z$, for positive real numbers x, y, z satisfying $x+y+z=\pi$. [*Ed.* compare problem 2524 [2001 : 157].] (Thus, x, y, z are the angles of some triangle.) Making this substitution, we get

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}} \;=\; \csc x+\csc y+\csc z\,.$$

But the function $f(x)=\csc x$ is convex on the interval $(0,\pi)$. (Note that $f''(x)=\dfrac{1+\cos^2 x}{\sin^3 x}>0$ for $0<x<\pi$.) Therefore, we may apply Jensen's Inequality to obtain

$$\csc x+\csc y+\csc z \;\geq\; 3\cdot\csc\left(\frac{x+y+z}{3}\right) \;=\; 3\csc\left(\frac{\pi}{3}\right) \;=\; 2\sqrt{3}\,.$$

The value of $2\sqrt{3}$ is actually attained when $x=y=z=\frac{\pi}{3}$; that is, when $a=b=c=\sqrt{3}$. Thus, the minimum value of our expression is $2\sqrt{3}$.

II. *Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.*

Letting $S(a,b,c)$ denote the expression that is to be minimized, we notice that

$$S(a,b,c) \;=\; \left|1+\frac{i}{a}\right|+\left|1+\frac{i}{b}\right|+\left|1+\frac{i}{c}\right|\,.$$

Applying the Triangle Inequality, we get

$$S(a,b,c) \;\geq\; \left|3+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)i\right| \;=\; \sqrt{9+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\,.$$

The given constraint $a+b+c=abc$ is equivalent to $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1$. Hence,

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= 2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \\ &= 3+\frac{1}{2}\left(\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2\right) \geq 3\,.\end{aligned}$$

Using this inequality above gives $S(a,b,c) \geq \sqrt{12} = 2\sqrt{3}$. Thus, the minimum value of $S(a,b,c)$ is $S(\sqrt{3},\sqrt{3},\sqrt{3}) = 2\sqrt{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER J. BRADLEY, Bristol, UK; JACQUES CHONÉ, Nancy, France; CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina (another solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KURT KNUEVEN, student, Northern Kentucky University, KY, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA (2 solutions); ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposers. Six solutions were either incorrect or incomplete.

Most solutions were similar to Solution I above. Some used a different initial substitution or no substitution at all, but most of these still used Jensen's Inequality. Several solutions proceeded by the method of Lagrange multipliers. These succeeded in finding the correct critical point, but failed to prove that this point corresponded to a global minimum subject to the given constraint. They were judged to be incomplete.

Murty observes that the numbers a, b, c need not be positive. This can be easily seen from Solution II, which remains valid as long as a, b, c are non-zero.

2815. [2003 : 111] Corrected. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that $\Gamma(O,R)$ is the circumcircle of $\triangle ABC$, where $\angle ACB \neq 60^\circ$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Construct equilateral triangles BCD and ACE such that A and D are on opposite sides of BC, and B and E are on opposite sides of AC.

(a) Show that CD and CE intersect Γ at fixed points F and G, respectively. Characterize these points.

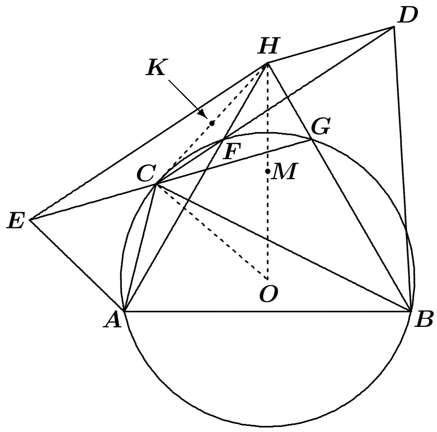
(b) Complete the parallelogram $DCEH$. Show that H is a fixed point. Characterize H.

(c) If K is the point of intersection of CH and DE, determine the locus of K as C varies.

[*Ed.* The problem as stated above incorporates a correction in part (c). Our featured solver made the correction and solved the corrected version.]

Solution by Titu Zvonaru, Bucharest, Romania.

Editor's comment. The solution makes use of directed angles. The symbol $\measuredangle XYZ$ represents the angle from line YX to line YZ. Alternatively, one can use undirected angles and analyze the cases that arise. The case presented below is valid for undirected angles when there is an acute angle at C, and the points lie on a circle in the order A, B, G, F, C.



(a) We first note that $\measuredangle BAF = \measuredangle BCF = \measuredangle BCD = 60^\circ$. We can also check that $\measuredangle GBA = \measuredangle ECA = 60^\circ$. Therefore, the points F and G are fixed: they are points on the circle Γ on the same side of AB as C such that the directed angles from BA to FA and from GB to AB are both 60°.

(b) We first show that $\triangle HDB \cong \triangle ACB$. We have $HD = EC = AC$ and $BD = BC$. Moreover,

$$\begin{aligned} \measuredangle HDB &= \measuredangle HDC + \measuredangle CDB \;=\; (180^\circ - \measuredangle DCE) + 60^\circ \\ &= 240^\circ - (360^\circ - \measuredangle ECA - \measuredangle ACB - \measuredangle BCD) \\ &= 240^\circ - (360^\circ - 60^\circ - 60^\circ - \measuredangle ACB) \;=\; \measuredangle ACB\,. \end{aligned}$$

Thus, by SAS, the triangles HDB and ACB are congruent. It follows that $HB = AB$. In a similar way, we obtain $HA = AB$. Therefore, H is fixed as the apex of the equilateral triangle with side AB located on the same side of AB as C.

(c) Let M be the mid-point of OH. Since K is the mid-point of CH, we have $MK = CO/2 = R/2$. Hence, the locus of K (as C moves on the arc of Γ from B to A) is the arc of the circle with centre M and radius $R/2$ between the mid-points of HB and HA.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2816. [2003 : 111] *Proposed by Boris Harizanov, student, Stara Zagora, Bulgaria.*

In acute-angled isosceles triangles $A_1B_1C_1$ (with $A_1C_1 = B_1C_1$) and $A_2B_2C_2$ (with $A_2C_2 = B_2C_2$), we have $A_1C_1 = A_2C_2$. For $k = 1, 2$, we have a circle with centre I_k and radius r_k inscribed in $\triangle A_kB_kC_k$, and a circle with centre O_k and radius R_k circumscribed around $\triangle A_kB_kC_k$.

If $I_1O_1 = I_2O_2$, is it true that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ must be congruent?

Solution by Christopher J. Bradley, Bristol, UK.

The answer is NO.

In a triangle with sides a, a and c, the circumradius R and the inradius r are given by $R = \dfrac{a^2}{\sqrt{4a^2-c^2}}$ and $r = \dfrac{c}{2}\sqrt{\dfrac{2a-c}{2a+c}}$. Thus,

$$OI^2 \;=\; R^2 - 2Rr \;=\; \frac{a^2(a-c)^2}{4a^2-c^2}\,.$$

If we have two triangles with sides a, a, c and a, a, d, then

$$\begin{aligned} O_1I_1^2 \;=\; O_2I_2^2 \quad&\Longrightarrow\quad \frac{(a-c)^2}{4a^2-c^2} \;=\; \frac{(a-d)^2}{4a^2-d^2} \\ &\Longrightarrow\quad c=d \quad\text{or}\quad (2c-5a)(2d-5a) \;=\; 9a^2\,. \end{aligned}$$

Thus, the triangles are not necessarily congruent.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Some solvers did not mention the case $OI = 0$ (but this editor has been kind to them). Zhou commented that the answer is, in fact, "YES" if IO means the **signed distance**.

2817. [2003 : 112] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that A, B, and C are the angles of $\triangle ABC$. Define

$$\begin{aligned} L &= 4\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\,; \\ M &= \left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right) \\ &\qquad\qquad \prod_{\text{cyclic}}\left(\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)-\cos\left(\frac{A}{2}\right)\right)\,. \end{aligned}$$

Show that $L = M$.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Firstly,

$$\begin{aligned} M &= \left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right) \\ &\quad \cdot\left(\cos^2\left(\frac{A}{2}\right)-\cos^2\left(\frac{B}{2}\right)-\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\right) \\ &= 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right) \\ &\qquad -\left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right)^2 . \end{aligned} \tag{1}$$

Since $\cos\left(\frac{B+C}{2}\right)=\sin\left(\frac{A}{2}\right)$, we have

$$\begin{aligned} &\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right) \\ &\qquad = \tfrac{1}{2}(\cos B+\cos C+2)-\left(1-\sin^2\left(\frac{A}{2}\right)\right) \\ &\qquad = \cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)+\sin^2\left(\frac{A}{2}\right) \\ &\qquad = \sin\left(\frac{A}{2}\right)\left(\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)\right) \\ &\qquad = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) . \end{aligned} \tag{2}$$

Substituting (2) into (1), we then have

$$M = 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\left(1-\sin^2\left(\frac{A}{2}\right)\right) = L .$$

II. *Solution by Arkady Alt, San Jose, CA, USA.*

Let $\alpha=\frac{\pi-A}{2}$, $\beta=\frac{\pi-B}{2}$, and $\gamma=\frac{\pi-C}{2}$. Then α, β, $\gamma>0$, and $\alpha+\beta+\gamma=\pi$. Thus, we can interpret α, β, γ as the angles of a triangle T. Without loss of generality, we may assume that T has circumradius $R=1/2$.

Let a, b, and c denote the sides of T. Then, by the Law of Sines, we have $a=\sin\alpha$, $b=\sin\beta$, and $c=\sin\gamma$. Note that

$$\cos\frac{A}{2} = \sin\left(\frac{\pi}{2}-\frac{A}{2}\right) = \sin\alpha = a .$$

Similarly, $\cos\left(\frac{B}{2}\right) = b$ and $\cos\left(\frac{C}{2}\right) = c$. Then, using the well-known formula $R=\frac{abc}{4K}$, where K denotes the area of T, we have

$$L = 4(abc)^2 = 64R^2K^2 = 16K^2 .$$

On the other hand, using Heron's Formula, we get

$$\begin{aligned} M &= (\sin\alpha+\sin\beta+\sin\gamma)\prod_{\text{cyclic}}(\sin\alpha+\sin\beta-\sin\gamma) \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) = 16K^2 . \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; JOSEPH LING, University of Calgary, Calgary, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2818. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that n, $k \geq 2$ are integers such that $(n + k^n, k) = 1$.

Prove that at least one of $n + k^n$ and $n\,k^{(k^n\,-\,1)} + 1$ is not prime.

Solution by Michel Bataille, Rouen, France.

Let $p = n + k^n$ and $q = nk^{(k^n-1)} + 1$. If p is composite, then we are done. Suppose instead that p is a prime. We are given that $(p, k) = 1$. By Fermat's Little Theorem, we have

$$q \;=\; (p - k^n)k^{p-n-1} + 1 \;\equiv\; -k^{p-1} + 1 \;\equiv\; 0 \pmod{p}\ .$$

Furthermore,

$$\begin{aligned} n\left(k^{(k^n-1)} - 1\right) \;&>\; k^{(k^n-1)} - 1 \;=\; \big(1 + (k-1)\big)^{(k^n-1)} - 1 \\ &\geq\; (k^n - 1)(k-1) \;\geq\; k^n - 1\,, \end{aligned}$$

and hence, $q = nk^{(k^n-1)} + 1 > n + k^n = p$. It follows that q is composite.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Most of the solutions are the same as the one featured above, except for the demonstration of the fact that $p < q$, which almost all solvers either took for granted or simply stated as "evident" or "clear". From the proof given above, it is obvious that the assumption "$(n + k^n, k) = 1$" is really superfluous. However, Parmenter is the only solver who explicitly pointed this out.

2819. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy, for all real x and y, $f\left(\dfrac{2x+y}{3}\right) \geq f\left(\sqrt[3]{x^2 y}\right)$.

Prove that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

Solution by Joseph Ling, University of Calgary, Calgary, AB, with minor modifications by the editor.

We prove that the only functions that have the given property are the constant functions. Indeed for any x,

$$f(x) \;=\; f\left(\frac{2(0)+(3x)}{3}\right) \;\geq\; f\left(\sqrt[3]{0^2(3x)}\right) \;=\; f(0)\,;$$

and by considering $y = \sqrt[3]{4}x$ and $z = -y/2$, we get

$$f(0) \;=\; f\left(\frac{2z+y}{3}\right) \;\geq\; f\left(\sqrt[3]{(-y/2)^2 y}\right) \;=\; f\left(\frac{y}{\sqrt[3]{4}}\right) \;=\; f(x)\,.$$

We remark that we can prove the claims in the proposed question if we just require that

$$f\left(\frac{2x+y}{3}\right) \;\geq\; f\left(\sqrt[3]{x^2 y}\right)$$

for all real numbers x and y such that $xy \geq 0$. Suppose that x, $y \in [0,\infty)$ with $x < y$. Consider the continuous function $h(t) = 2t^3 - 3yt^2 + x^3$. Since $h(0) = x^3 \geq 0 > -y^3 + x^3 = h(y)$, there is some $u \in [0,y)$ such that $h(u) = 0$, by Intermediate Value Theorem. Thus, $2u^3 - 3yu^2 + x^3 = 0$, and hence, $x^3 = u^2(3y-2u)$. Letting $v = 3y - 2u$, we have $v > 0$ and $x^3 = u^2 v$. Therefore,

$$f(y) \;=\; f\left(\frac{2u+v}{3}\right) \;\geq\; f\left(\sqrt[3]{u^2 v}\right) \;=\; f(x)\,.$$

This proves that f is increasing on $[0,\infty)$.

Let $g(x) = f(-x)$. For all x, y with $xy \geq 0$, we have

$$\begin{aligned} g\left(\frac{2x+y}{3}\right) \;&=\; f\left(\frac{2(-x)+(-y)}{3}\right) \;\geq\; f\left(\sqrt[3]{(-x)^2(-y)}\right) \\ &=\; f\left(-\sqrt[3]{x^2 y}\right) \;=\; g\left(\sqrt[3]{x^2 y}\right)\,. \end{aligned}$$

It follows that g is increasing on $[0,\infty)$, and thus, f is decreasing on $(-\infty,0]$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Guersenzvaig also recognized that f is constant, and that the claims hold under the weaker hypothesis stated above. Other solvers who proved f is constant are Arslanagić, Bataille, Furdui, and Janous.

2820. [2003 : 113] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that Q is any point in the plane of $\triangle ABC$. Suppose that AQ, BQ, CQ meet BC, CA, AB at D, E, F, respectively; that L, M, N are the mid-points of BC, CA, AB, respectively; and that U, V, W are the mid-points of AQ, BQ, CQ, respectively.

It is known that a conic Σ passes through D, E, F, L, M, N, U, V, and W. Clearly, if Σ is enlarged by a factor of 2, with Q as the centre of enlargement, then the resulting conic Σ_Q passes through A, B, and C.

Suppose that P is any point on Σ_Q, and that lines through P parallel to AQ, BQ, CQ meet the sides BC, CA, AB at R, S, T, respectively.

Prove that R, S, and T are collinear.

Combination of solutions by David Loeffler, student, Trinity College, Cambridge, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

We show that the collinearity follows quickly from the affine versions of two familiar Euclidean theorems. We can apply a linear transformation to the plane so that A, B, C, and Q are mapped to a triangle and its orthocentre: apply a shear parallel to BC until $AQ \perp BC$, then stretch the plane parallel to AQ until $BQ \perp AC$. (Compare the solution to 2822 and its reference [2].) Since the problem concerns only affinely invariant properties, it is sufficient to prove the result for the case where Q is the orthocentre of triangle ABC (so that AQ, BQ, CQ are altitudes). In this case the conic Σ through D, E, F, L, M, N, U, V, W is the nine-point circle of triangle ABC, and Σ_Q is the circumcircle. Note that for any point P in the plane, PR, PS, and PT are lines perpendicular to the sides (since they are assumed to be parallel to the altitudes). The feet of these perpendiculars, namely R, S, and T, are collinear (on the Simson line) if and only if P lies on the circumcircle of $\triangle ABC$. Since we are given that P lies on Σ_Q, we conclude that R, S, and T are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

2821. [2003 : 113] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In triangle $\triangle ABC$, let w_a, w_b, w_c be the lengths of the interior angle bisectors, and r the inradius. Prove that

$$\frac{1}{{w_a}^2}+\frac{1}{{w_b}^2}+\frac{1}{{w_c}^2} \;\le\; \frac{1}{3r^2}\,,$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we can assume that $a \le b \le c$. Since $w_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$, $w_b = \dfrac{2\sqrt{cas(s-b)}}{c+a}$, $w_c = \dfrac{2\sqrt{abs(s-c)}}{a+b}$, and $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, the given inequality is equivalent to

$$\begin{aligned} 4abc(a+b+c)^2 - 3a(b+c)^2(a+b-c)(a-b+c) \\ - 3b(c+a)^2(b+c-a)(b-c+a) \\ - 3c(a+b)^2(c+a-b)(c-a+b) \ \ge\ 0\,. \end{aligned}$$

The left side above is equal to

$$\tfrac{1}{4}\Big((b-c)^2\Big[11ab(b-a) + 11ac(c-a) + 12bc(b+c) + 10abc - 4a^3\Big] \\ + a(b+c)(3a+b+c)(2a-b-c)^2\Big)\,,$$

which is clearly non-negative. Thus, the inequality is true. Equality holds if and only if $a = b = c$.

Note that this proof does not require that a, b, and c are the sides of a triangle, as long as they are non-negative.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; TITU ZVONARU, Bucharest, Romania; and the proposer. There were also two incorrect solutions submitted.

This solution will probably leave our readers wondering if there is a general method behind the grouping and the factorization in Zhou's proof (see also his proof of Crux problem 2807). He claims that there is such a method and challenges the readers to figure it out.

The proposer has also asked the more general question: What is the set of all exponents p such that

$$\frac{1}{w_a^p} + \frac{1}{w_b^p} + \frac{1}{w_c^p} \ \le\ \frac{1}{3^{p-1}r^p}\ ?$$

2822. [2003 : 114] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Suppose that Π is a parallelogram with sides of lengths $2a$ and $2b$ and with acute interior angle α, and that F and F' are the foci of the ellipse Λ that is tangent to the four sides of Π at their mid-points.

(a) Find the major and minor semi-axes of Π in terms of a, b, and α.

(b) Find a straight-edge and compass construction for F and F'.

Solution by Michel Bataille, Rouen, France, with added explanations by the editor.

(a) Let $\dfrac{x^2}{u^2}+\dfrac{y^2}{v^2}=1$ be the equation of Λ, where $u>v>0$. Let T_j ($j=1,2,3,4$) be the mid-points of the sides of Π (see Figure 1).

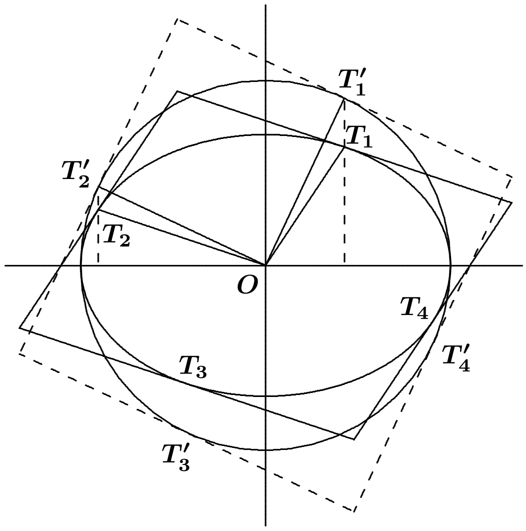


Figure 1

Consider the normal perspective affinity $\mathcal{A}$ whose axis is the x–axis and whose scale factor is u/v. (*Normal* implies that the strain is in the direction of the y–axis; see reference [2] for an elementary and pleasant treatment of perspective affinities and their link to ellipses.) The ellipse Λ is transformed by $\mathcal{A}$ into its principal circle Λ' (centre O, radius u). Since $\mathcal{A}$ preserves parallelism, mid-points, and contacts, $\mathcal{A}$ transforms Π into a parallelogram Π' whose sides touch Λ' at their mid-points $T_j'=\mathcal{A}(T_j)$ ($j=1,2,3,4$). As such, Π' is a square.

Now, let θ be a real number such that $T_1'=(u\cos\theta,u\sin\theta)$ (which implies that $T_1=(u\cos\theta,v\sin\theta)$). Then, since $OT_1'\perp OT_2'$, we have

$$T_2=\left(u\cos(\theta+\tfrac{\pi}{2}),v\sin(\theta+\tfrac{\pi}{2})\right)=(-u\sin\theta,v\cos\theta)\,.$$

It follows that

$$\begin{aligned} a^2 &= OT_1^2 = u^2\cos^2\theta+v^2\sin^2\theta\,,\\ b^2 &= OT_2^2 = u^2\sin^2\theta+v^2\cos^2\theta\,. \end{aligned}$$

Moreover, the area of the parallelogram with sides $\overrightarrow{OT_1}$ and $\overrightarrow{OT_2}$ is $OT_1\cdot OT_2\sin\alpha=ab\sin\alpha$. Since this area is a quarter the area of Π, and since, for any ellipse, all parallelograms that are tangent at their mid-points have the same area, it follows that $uv=ab\sin\alpha$. From these relations we obtain first $a^2+b^2=u^2+v^2$, then $u\pm v=(a^2+b^2\pm 2ab\sin\alpha)^{1/2}$, and finally

$$\begin{aligned} u &= \tfrac{1}{2}\left((a^2+b^2+2ab\sin\alpha)^{1/2}+(a^2+b^2-2ab\sin\alpha)^{1/2}\right) , \\ v &= \tfrac{1}{2}\left((a^2+b^2+2ab\sin\alpha)^{1/2}-(a^2+b^2-2ab\sin\alpha)^{1/2}\right) . \end{aligned}$$

(b) Given Π (and the mid-points T_j, ($j=1$, 2, 3, 4) of its sides), we will construct the axes and vertices of Λ. The foci are then readily obtained. First, draw the circle Γ with diameter T_2T_4, and denote by S_1, S_3 the points of intersection of Γ with the perpendicular to T_2T_4 at the centre O of Π. The circle centred on the line T_2T_4 and passing through S_1 and T_1 meets T_2T_4 at U, V. (See Figure 2.)

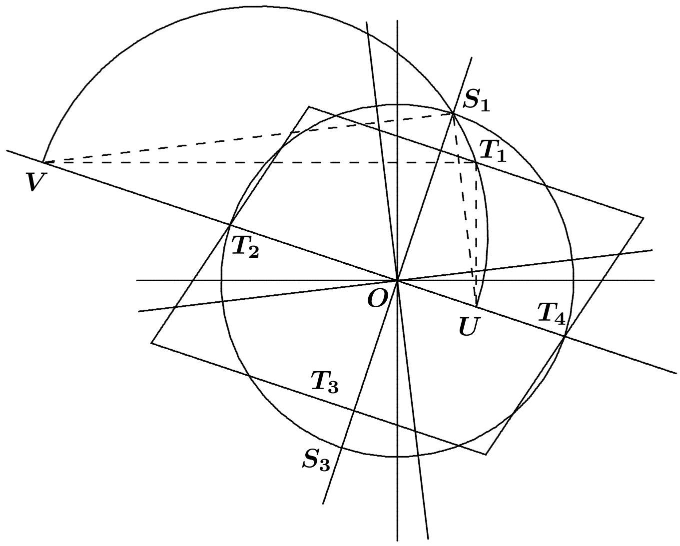


Figure 2

Let $\mathcal{B}$ be the perspective affinity with axis T_2T_4 that transforms S_1 into T_1, and S_3 into T_3. Claim: $\mathcal{B}$ transforms Γ into our desired conic Λ. [$\mathcal{B}$ takes Γ into a conic through T_1, T_2, T_3, T_4. We must show, therefore, that the sides of the given parallelogram are the images under $\mathcal{B}$ of tangents to Γ. The tangent to Γ at S_1 is taken by $\mathcal{B}$ to the line parallel to it through T_1; but that line must be the side of the parallelogram through T_1, since that side is perpendicular to OS_1, as is the tangent to Γ through S_1. As for T_2, it is a fixed point; thus, the tangent to Γ at T_2, which is parallel to OS_1, must be taken into the line through T_2 that is parallel to OT_1, which is a side of the parallelogram, as claimed.]

Note that since UV is a diameter of Γ, US_1 and VS_1 are perpendicular, as are their images UT_1 and VT_1. This provides two perpendicular directions transformed by $\mathcal{B}$ into two perpendicular directions. It follows that the axes of Λ are the lines through O that are parallel to UT_1 and VT_1. [The lines through O that are parallel to US_1 and VS_1 are perpendicular, and therefore conjugate with respect to Γ. These are taken into two lines through O that are conjugate with respect to Λ; these two lines are also perpendicular. The axes of a conic are the two lines through its centre that are both conjugate and perpendicular.] Consequently, the vertices of Λ must be the images under $\mathcal{B}$ of the points of intersection of Γ with the diameters parallel to US_1 and VS_1. (See Figure 3, where, for practical reasons, S_3 and its image T_3 have been used instead of S_1 and T_1 in constructing A, a vertex on the major axis, and B, a vertex on the minor axis.) Finally, note that the foci are the intersection points of line OA with the circle centred at B whose radius is OA [since the distance c from O to the foci satisfies $c^2 = u^2 - v^2 = OA^2 - OB^2$].

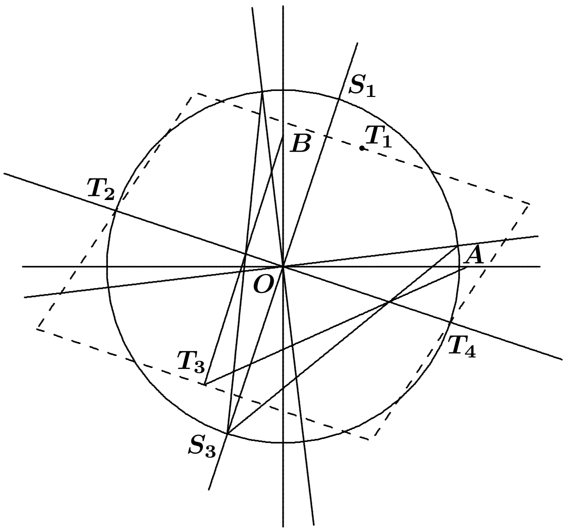


Figure 3

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (part (a) *only); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

In [3] *Dan Pedoe shows how to construct the centre and foci of an ellipse given only five of its points. Konečný provided, instead of a construction, reference* [1] *where it is shown how to inscribe an ellipse in a given parallelogram with tangency at a prescribed point of one side. He also recommends the essay by Naoki Sato* [4].

References

[1] Heinrich Dörrie, *100 Great problems of Elementary Mathematics*. Dover, 1965. (German title: *Triumph der Mathematik*.)

[2] Max Jeger, *Transformation Geometry*. Allen and Unwin Ltd., 1966.

[3] Dan Pedoe, Pascal Redivivus II, *Crux Math.* [1979 : 281–287]

[4] Naoki Sato, Ellipses in polygons. *Crux with Mayhem* [2000 : 361–371].

2823. [2003 : 114] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that L, M, N are points on BC, CA, AB, respectively, and are distinct from A, B and C. Suppose further that

$$\frac{BL}{LC} = \frac{1-\lambda}{\lambda}, \quad \frac{CM}{MA} = \frac{1-\mu}{\mu}, \quad \text{and} \quad \frac{AN}{NB} = \frac{1-\nu}{\nu},$$

and that the circles AMN, BNL, and CLM meet at the Miquel point P.

Find $[BCP] : [CAP] : [ABP]$ in terms of λ, μ, ν and the side lengths of $\triangle ABC$.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We will solve the problem using barycentric (areal) coordinates with ABC as the triangle of reference.

The barycentric coordinates of the given points in this problem are

$$A(1,0,0), \quad B(0,1,0), \quad C(0,0,1),$$
$$L(0,\lambda,1-\lambda), \quad M(1-\mu,0,\mu), \quad N(\nu,1-\nu,0).$$

In order to solve the problem, we need the barycentric coordinates of the point P, since these are proportional to the areas $[BCP]$, $[CAP]$, and $[ABP]$.

It is known that, in barycentric coordinates (x,y,z), the equation of a circle has the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0, \qquad (1)$$

where a, b, c are the sides of $\triangle ABC$ opposite the vertices A, B, C, respectively, and p, q, r are the powers of the points A, B, C, respectively, with respect to the circle. We note in passing that the equation of the circumcircle of ABC is simply

$$a^2yz + b^2zx + c^2xy = 0.$$

(As a reference for these results, see, for instance, the article in the digital journal FORUM GEOMETRICORUM with URL

`http://www.math.fau.edu/yiu/clawson.pdf.`

This article is about the so-called "Clawson Point" of a triangle, referring to an old problem in ***Crux Mathematicorum*** [1983 : 23–24]. The article attributes the result (1) to John Conway.)

First, we will obtain the equation of the circle CLM. We use (1) to make the following conclusions: Since the circle passes through $C(0,0,1)$, we have $r=0$; since the circle passes through $L(0,\lambda,1-\lambda)$, we obtain $q=a^2(1-\lambda)$; since the circle passes through $M(1-\mu,0,\mu)$, we get $p=b^2\mu$. Therefore, the equation of circle CLM is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\big(b^2\mu x + a^2(1-\lambda)y\big) = 0. \qquad (2)$$

Analogously, the equation of circle BNL is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\bigl(c^2(1-\nu)x + a^2\lambda z\bigr) \;=\; 0\,. \qquad (3)$$

These two circles intersect at P (and at L). By solving equations (2) and (3), perhaps with the help of a computer algebra system such as MAPLE, it can be shown that the barycentric coordinates of P are

$$\begin{aligned} x &= a^2\bigl[-a^2\lambda(1-\lambda) + b^2(1-\lambda)(1-\mu) + c^2\lambda\mu\bigr]\,, \\ y &= b^2\bigl[a^2\lambda\mu - b^2\mu(1-\mu) + c^2(1-\mu)(1-\nu)\bigr]\,, \\ z &= c^2\bigl[a^2(1-\lambda)(1-\nu) + b^2\mu\nu - c^2\nu(1-\nu)\bigr]\,. \end{aligned}$$

The ratio of these coordinates is $[BCP] : [CAP] : [ABP]$.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let R, r, s, and t be the circumradii of $\triangle ABC$, $\triangle AMN$, $\triangle BNL$, and $\triangle CLM$, respectively. Let $x = MN$, $y = NL$, $z = LM$, $u = LP$, $v = MP$, $w = NP$, and $\theta = \angle AMP = \angle BNP = \angle CLP$. Applying Ptolemy's Theorem in the cyclic quadrilaterals $CLPM$, $AMPN$, and $BNPL$, we get

$$\begin{aligned} \begin{bmatrix} (1-\mu)b & \lambda a & 0 \\ 0 & (1-\nu)c & \mu b \\ \nu c & 0 & (1-\lambda)a \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} CP\cdot z \\ AP\cdot x \\ BP\cdot y \end{bmatrix} \\ &= 2\sin\theta \begin{bmatrix} tz \\ rx \\ sy \end{bmatrix}\,. \end{aligned}$$

By Cramer's Rule, the ratio $u:v:w$ is given by $D_1 : D_2 : D_3$, where

$$D_1 = \begin{vmatrix} tz & \lambda a & 0 \\ rx & (1-\nu)c & \mu b \\ sy & 0 & (1-\lambda)a \end{vmatrix}\,, \quad D_2 = \begin{vmatrix} (1-\mu)b & tz & 0 \\ 0 & rx & \mu b \\ \nu c & sy & (1-\lambda)a \end{vmatrix}\,,$$

$$\text{and } D_3 = \begin{vmatrix} (1-\mu)b & \lambda a & tz \\ 0 & (1-\nu)c & rx \\ \nu c & 0 & sy \end{vmatrix}\,.$$

By the Sine Law, $\frac{z}{t} = 2\sin C = \frac{c}{R}$. Thus, $\frac{tz}{R} = \frac{z^2}{c} = n$, where

$$\begin{aligned} n &= \frac{\lambda^2a^2 + (1-\mu)^2b^2 - 2\lambda(1-\mu)ab\cos C}{c} \\ &= \frac{\lambda^2a^2 + (1-\mu)^2b^2 - \lambda(1-\mu)(a^2+b^2-c^2)}{c}\,, \end{aligned}$$

by the Cosine Law. Likewise, we have

$$\begin{aligned} \frac{rx}{R} &= \ell = \frac{\mu^2b^2 + (1-\nu)^2c^2 - \mu(1-\nu)(b^2+c^2-a^2)}{a}\,, \\ \frac{sy}{R} &= m = \frac{\nu^2c^2 + (1-\lambda^2)a^2 - \nu(1-\lambda)(c^2+a^2-b^2)}{b}\,. \end{aligned}$$

Hence, $[BCP] : [CAP] : [ABP] = au : bv : cw = T_1 : T_2 : T_3$, where

$$T_1 = a\begin{vmatrix} n & \lambda a & 0 \\ \ell & (1-\nu)c & \mu b \\ m & 0 & (1-\lambda)a \end{vmatrix},$$

$$T_2 = b\begin{vmatrix} (1-\mu)b & n & 0 \\ 0 & \ell & \mu b \\ \nu c & m & (1-\lambda)a \end{vmatrix},$$

$$T_3 = c\begin{vmatrix} (1-\mu)b & \lambda a & n \\ 0 & (1-\nu)c & \ell \\ \nu c & 0 & m \end{vmatrix}.$$

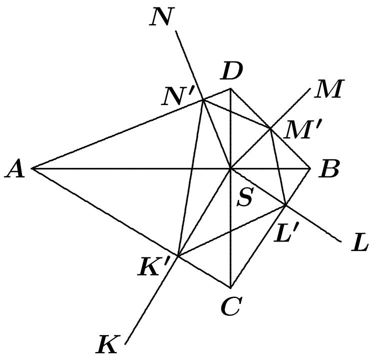
Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

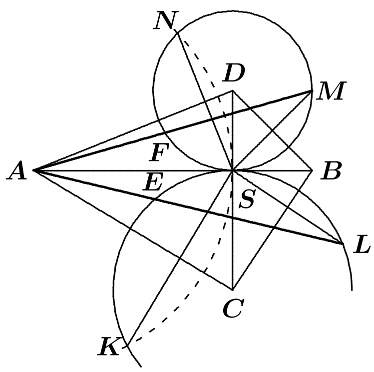
2824. [2003 : 115] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Two perpendicular line segments AB and CD intersect at S. Denote by K, L, M, N the reflections of S in the lines AC, BC, BD, AD, respectively. Suppose that the circumcircle of $\triangle SKL$ meets the line AL again in E, and that the circumcircle of $\triangle SMN$ meets the line AM again in F.

Prove that quadrilateral $KEFN$ is cyclic.

Solution by John G. Heuver, Grande Prairie, AB.





Let SK, SL, SM, SN intersect AC, BC, BD, AD in K', L', M', N', as in the diagram above. [Because of the right angles at K' and N', SA is a diameter of circle $SK'AN'$, with analogous statements for the circles having diameters SB, SC, and SD.] It follows, as shown in the diagram, that

$$\begin{aligned} \angle L'K'N' + \angle N'M'L' &= \angle L'K'S + \angle SK'N' + \angle N'M'S + \angle SM'L' \\ &= \angle BCS + \angle SAD + \angle ADS + \angle SBC \;=\; \pi\,. \end{aligned}$$

This implies that quadrilateral $K'L'M'N'$ is cyclic; hence, quadrilateral $KLMN$ is cyclic, since its sides are parallel to the former. Further, because AD and BD are perpendicular bisectors of SN and SM, it follows that D is the center of circle SNM. Similarly, C is the center of circle SKL, from which follows that AB is tangent to both circles. We have $AS = AK = AN$. The circle with centre A and radius AS inverts circles SNM and SKL into themselves, since they are orthogonal to the circle of inversion. Points L and M invert into E and F, respectively, while points K and N are invariant. Thus the circumcircle of $KLMN$ is inverted into the circumcircle of quadrilateral $KEFN$. More precisely, the points K, E, F, N lie on a circle, or (should circle $KLMN$ contain A) they lie on a line.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

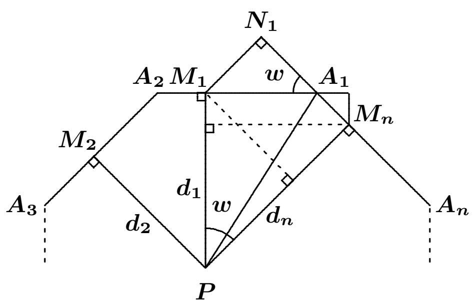
2825★. [2003 : 115] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\mathcal{R}_n$ be a regular n–gon ($n \geq 3$), and let $\mathcal{P}_n$ be the set of all points P in $\mathcal{R}_n$ such that all n perpendiculars from P to the sides of $\mathcal{R}_n$ have feet lying in the interior of the respective sides. These feet, the endpoints of the respective sides, and the point P form $2n$ (right-angled) triangles. Let S_1 and S_2 be the sums of the areas of n triangles each, using alternate triangles.

Prove that $S_1 = S_2$ for all points P in $\mathcal{P}_n$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let d_1, d_2, …, d_n be the distances PM_1, PM_2, …, PM_n, respectively, from P to the sides of the polygon (with M_i on A_iA_{i+1}, where the subscripts are taken modulo n). Let N_1 be the orthogonal projection of M_1 onto the side A_1A_n, and let $w = \angle M_1A_1N_1 = \angle M_1PM_n$ be the external angle of the regular n–gon.



We have

$$A_1M_1 \;=\; M_1N_1 \csc w \;=\; (d_n - d_1 \cos w) \csc w \;=\; d_n \csc w - d_1 \cot w \,,$$

and

$$2 \times \text{Area}\ (PA_1M_1) \;=\; d_1 \cdot A_1M_1 \;=\; d_nd_1 \csc w - d_1^2 \cot w\,.$$

Hence, going counterclockwise, we get

$$2S_1 \;=\; (d_nd_1 + d_1d_2 + \cdots + d_{n-1}d_n) \csc w - (d_1^2 + d_2^2 + \cdots + d_n^2) \cot w\,.$$

Similarly, by projecting M_n on the side A_1A_2, we have

$$2 \times \text{Area}\ (PA_1M_n) \;=\; d_n \cdot A_1M_n \;=\; d_nd_1 \csc w - d_n^2 \cot w\,,$$

and, going clockwise, we get

$$2S_2 \;=\; (d_nd_1 + d_{n-1}d_n + \cdots + d_1d_2) \csc w - (d_n^2 + d_{n-1}^2 + \cdots + d_1^2) \cot w\,.$$

Thus, $S_1 = S_2$.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Our problem bears a superficial resemblance to a "proof without words" from ten years ago ["Proof Without Words: Fair Allocation of a Pizza" by Larry Carter and Stan Wagon, Math. Mag. **67**:4 *(Oct. 1994) p. 267*]:

> If a pizza is divided into $2n$ slices by making cuts at angles of π/n from an arbitrary point P in the pizza, then the sums of the areas of alternate slices are equal when n is even and greater than 2. For general P the sums are not equal when n is 2 or odd.

The authors provide a partial proof and several references, including Crux problem 1325 [1989 : 120–122].

Most solvers of our problem determined the area of S_1 explicitly, showing it to be half the area of the polygon. Loeffler pointed out two worthy by-products that follow from such an approach: (1) *for any point P in a regular n–gon centred at O, the centroid of the pedal n–gon is the mid-point of OP; and* (2) *$S_1 = S_2$ for any point P in the plane, provided that we use signed areas. When Loeffler's generalization is applied to a polygonal pizza, however, even a pure mathematician might object if his portion of the pizza were to contain too much negative area.*