Pólya's Paragon

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For this month's installment, I have decided to explore some of the curious and interesting properties of sequences and series. To begin, I would like to revisit a famous problem that is said to have been solved by the famous mathematician Gauss when he was very young. The story goes that his teacher was frustrated with how quickly he could solve the problems given out in class. Therefore, he was assigned to sum the numbers from 1 to 100, to keep him busy. Remarkably, in almost no time at all, he had solved the problem, much to the teacher's amazement.

Here is the trick he is said to have used:

$$\begin{array}{rcccccc} S &=& 1 &+& 2 &+\cdots+& 100\,, \\ S &=& 100 &+& 99 &+\cdots+& 1\,, \\ 2S &=& 101 &+& 101 &+\cdots+& 101\,, \\ 2S &=& 101\cdot 100\,, \\ S &=& 5050\,. \end{array}$$

By writing the terms forward and backward, we are able to get a very nice expression for twice the sum. The third line is the result of summing the first two lines term by term. Since we know that there are exactly 100 terms, we quickly arrive at the answer.

We would like to be able to use this trick for finding other sums as well. By generalizing, we will now call this a 'technique' which we can use for all sorts of other situations. This time, we will start with an arithmetic sequence where the first term is a, the terms increase by d, and there are n terms. Here is what happens:

$$\begin{array}{rcccccc} S &=& a &+& a+d &+\cdots+& \big(a+(n-1)d\big)\,, \\ S &=& \big(a+(n-1)d\big) &+& \big(a+(n-2)d\big) &+\cdots+& a\,, \\ 2S &=& \big(2a+(n-1)d\big) &+& \big(2a+(n-1)d\big) &+\cdots+& \big(2a+(n-1)d\big)\,, \\ 2S &=& n\big(2a+(n-1)d\big)\,, \\ S &=& na+\dfrac{n(n-1)}{2}d\,. \end{array}$$

We can use this formula to determine that the sum of the first n natural numbers is exactly $n(n+1)/2$. To see this, use $a=1$ and $d=1$ in the previous equation.

We might now ask ourselves what sort of sums we can achieve when the terms do not form an arithmetic progression. Here are a few more sums with interesting patterns that I will present without proof:

$$\begin{aligned}\frac{n(n+1)}{2} &= 1+2+\cdots+n\,,\\ \frac{n(n+1)(n+2)}{3} &= 1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)\,,\\ \frac{n(n+1)(n+2)(n+3)}{4} &= 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots\\ &\qquad +n\cdot(n+1)\cdot(n+2)\,.\end{aligned}$$

We can use these identities to discover even more sums, like the sum of squares shown here:

$$\begin{aligned}1^2+2^2+\cdots+n^2 &= \big(1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)\big)\\ &\qquad -(1+2+\cdots+n)\\ &= \frac{n(n+1)(n+2)}{3}-\frac{n(n+1)}{2}\\ &= \frac{n(n+1)(2n+1)}{6}\,.\end{aligned}$$

Finally, I would like to look at numbers called ‘triangular’ numbers. The k^{th} triangular number is the sum of the first k natural numbers. The first five triangular numbers are 1, 3, 6, 10, and 15. Is there an easy way to find the sum of the first n triangular numbers? The answer is yes! Even though they do not form an arithmetic sequence, we can still find their sum.

$$\begin{aligned}1+3+6+\cdots+\frac{n(n+1)}{2} &= \frac{1^2+1}{2}+\frac{2^2+2}{2}+\cdots+\frac{n^2+n}{2}\\ &= \frac{1}{2}\left(1^2+2^2+\cdots+n^2\right)\\ &\qquad +\frac{1}{2}\left(1+2+\cdots+n\right)\\ &= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{1}{2}\left(\frac{n(n+1)}{2}\right)\\ &= \frac{n(n+1)(n+2)}{6}\,.\end{aligned}$$

Now that you know some useful techniques and identities for finding sums, here are a couple of problems for you to try:

1. Find the sum of the first n cubes. That is, find $1^3 + 2^3 + \cdots + n^3$.
2. Find a relationship between your result from problem 1 and one of the other identities used in this article.
3. Find the sum of the reciprocals of the triangular numbers. That is, find $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \cdots$

   HINT: This is an infinite sum. Start by looking at half this sum, and write each term as the difference of two fractions.