On Some Examples of Geometric Fallacies

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Euclid wrote a book containing a collection of geometric fallacies called Pseudaria. But this book was lost. Since Euclid's time many amusing examples of geometric fallacies have been published. Two well-known examples are the following ones that appeared in Rouse Ball's *Mathematical Recreations and Essays*.

(1) To prove that a right angle is equal to an angle that is greater than a right angle.

(2) To prove that every triangle is isosceles.

In these examples, when the figures are accurately drawn, the mistakes quickly become apparent.

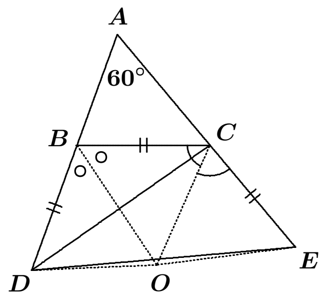
Even when the figures are accurately drawn and the argument is correct, geometric fallacies may occur. We often deduce false conclusions for lack of careful consideration. If the false conclusion is not absurd, it can easily be overlooked.

We propose some new examples of geometric fallacies. Errors in the fallacious geometric proofs will be explained afterwards in the Answer Section.

First we consider a (true) theorem, and then we investigate its converse. As is well known, the converse of a theorem is not always true. Geometric fallacies happen quite frequently in the attempted proof of the converse.

Theorem 1 Let ABC be a triangle with $\angle A = 60^\circ$. Let D be the point on AB produced beyond B such that $BD = BC$, and let E be the point on AC produced beyond C such that $BC = CE$. Then $\angle DBC = 2\angle CED$, $\angle BCE = 2\angle BDE$, and $\angle CDE = 30^\circ$.

Proof. Let O be the intersection of the bisectors of $\angle DBC$ and $\angle BCE$. Since $BD = BC = CE$, we have $\triangle BDO \equiv \triangle BCO$ and $\triangle BCO \equiv \triangle ECO$. Thus, $\angle BDO = \angle BCO$, $\angle BOD = \angle BOC$, $DO = CO$, and also $\angle CBO = \angle CEO$, $\angle BOC = \angle COE$. Since $\angle DBO = \angle CBO$ and $\angle BCO = \angle ECO$, we have



$$\begin{aligned}\angle BOC &= 90^\circ - \frac{1}{2}\angle BAC \\ &= 90^\circ - 30^\circ = 60^\circ .\end{aligned}$$

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Hence, $\angle DOE = 3\angle BOC = 180^\circ$. Therefore, D, O, E are collinear. Thus,

$$\angle DBC = 2\angle CBO = 2\angle CEO = 2\angle CED$$

and

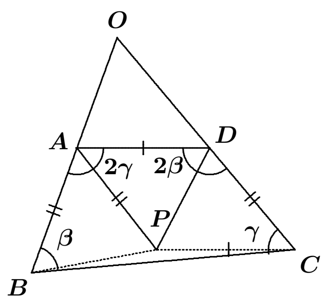
$$\angle BCE = 2\angle BCO = 2\angle BDO = 2\angle BDE .$$

Because $DO = CO$, we see that $\angle ODC = \angle OCD$. Therefore, $\angle CDE = \angle CDO = \frac{1}{2}\angle COE = 30^\circ$.

Next we shall consider two converses of this theorem.

Example 1. Let $ABCD$ be a convex quadrilateral with $AB = CD$. If we have $\angle A = 2\angle C$ and $\angle D = 2\angle B$, then $AB = AD$.

Proof. Let $\beta = \angle B$ and $\gamma = \angle C$. Then $\angle A = 2\gamma$ and $\angle D = 2\beta$. Since $\angle A + \angle B + \angle C + \angle D = 360^\circ$, we have $2\gamma + \beta + \gamma + 2\beta = 360^\circ$. Therefore, $\beta + \gamma = 120^\circ$. Let O be the point of intersection of AB and CD. Then



$$\angle BOC = 180^\circ - (\beta + \gamma) = 60^\circ .$$

Let P be the point such that $AP \parallel DC$ and $CP \parallel DA$. Since $APCD$ is a parallelogram, we have $AP = DC = AB$, and $PC = AD$, $\angle APC = \angle ADC = 2\beta$. Since $\angle BAP = \angle BOC = 60^\circ$ and $AB = AP$, we see that $\triangle ABP$ is equilateral. Thus, $\angle ABP = \angle APB = 60^\circ$. Since

$$\angle PBC = \angle ABC - \angle ABP = \beta - 60^\circ$$

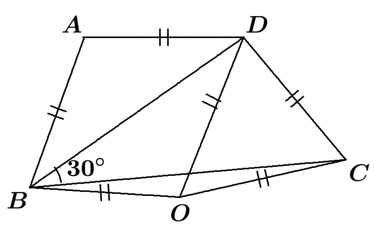
and

$$\angle BPC = 360^\circ - (\angle APB + \angle APC) = 360^\circ - (60^\circ + 2\beta) ,$$

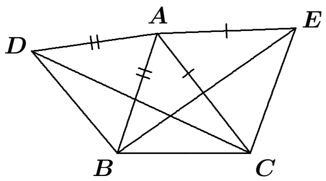
we have $\angle PCB = 180^\circ - (\angle PBC + \angle BPC) = \beta - 60^\circ$. Therefore, $\angle PBC = \angle PCB$, so that $PB = PC$. This implies $AB = AD$.

Example 2. Let $ABCD$ be a convex quadrilateral with $AB = AD = CD$. If $\angle DBC = 30^\circ$, then $\angle A = 2\angle C$.

Proof. Let O be the circumcentre of $\triangle BCD$. Then $OB = OC = OD$ and we have $\angle DOC = 2\angle DBC = 60^\circ$. Therefore, $\triangle OCD$ is equilateral, which means that $OB = OD = OC = CD$. Since $AB = AD = CD$, it follows that $AB = AD = OD = OB$; that is, $ABOD$ is a rhombus. Furthermore, $\angle BAD = \angle BOD = 2\angle BCD$.



Example 3. Suppose triangle ABC is given and equilateral triangles ABD and ACE are drawn outwardly on the sides AB and AC. If D, B, C, E are concyclic, then $AB = AC$.



Proof. Since D, B, C, E are concyclic, we see that $\angle BDC = \angle BEC$. Since $\angle BDA = \angle CEA = 60°$, it follows that

$$\begin{aligned}\angle ADC &= \angle BDA - \angle BDC \\ &= \angle CEA - \angle BEC \\ &= \angle AEB\,. \end{aligned} \tag{1}$$

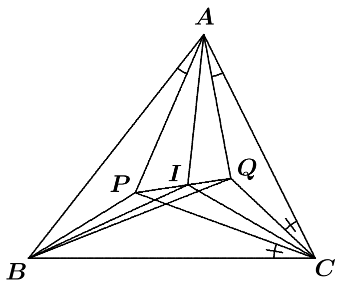
Since $AD = AB$, $AC = AE$, and $\angle DAC = \angle BAE$, it follows that $\triangle ADC \equiv \triangle ABE$, so that

$$\angle ADC = \angle ABE \tag{2}$$

From (1) and (2) we have $\angle AEB = \angle ABE$. Consequently, $AB = AE$. Since $AC = AE$, we have $AB = AC$.

Example 4. Suppose P and Q are two interior points of triangle ABC such that $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$. Suppose further that $AP : AQ = CP : CQ$. Then $BP : BQ = AP : AQ$.

Proof. Since $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$, we know that P and Q are isogonally conjugate points of $\triangle ABC$. Hence, $\angle ABP = \angle CBQ$. Let I be the intersection of PQ with the bisector of $\angle PAQ$. Then $PI : IQ = AP : AQ = CP : CQ$. Thus, $\angle PCI = \angle QCI$. Since $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$, we have



$$\angle BAI = \angle PAB + \angle PAI = \angle QAC + \angle QAI = \angle CAI\,.$$

Similarly, we have $\angle BCI = \angle ACI$. Thus, I is the incentre of $\triangle ABC$, so that $\angle ABI = \angle CBI$. Therefore,

$$\angle PBI = \angle ABI - \angle ABP = \angle CBI - \angle CBQ = \angle QBI\,.$$

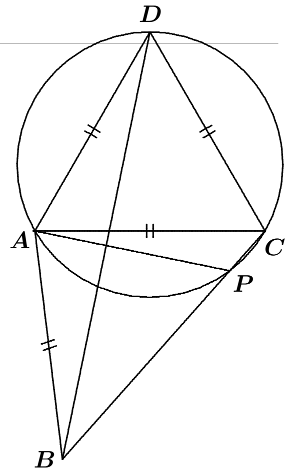
Hence, $BP : BQ = PI : IQ = AP : AQ$.

Remark. Notice in the above figure that $\dfrac{AP}{AQ} = \dfrac{CP}{CQ} > 1$, but $\dfrac{BP}{BQ} < 1$.

ANSWERS. The errors in the fallacious geometric proofs are briefly explained as follows.

Example 1. The mistake lies in the argument that if $\angle PBC = \angle PCB$, then $PB = PC$. If $\angle PBC = \angle PCB = 0°$, we cannot conclude that

$PB = PC$. Counterexample: suppose $ABCD$ is an isosceles trapezoid with $AB = CD \neq AD$ and $\angle A = \angle D = 120^\circ$.



Example 2. The mistake lies in the argument that we tacitly assumed that A and O are distinct points. There exists a case where A is the circumcenter of $\triangle BCD$. In this case, major angle BAD is equal to $2\angle BCD$. Counterexample: suppose $\triangle ACD$ is equilateral, and P is a point on the minor arc AC of the circumcircle of $\triangle ACD$ such that $AP > PC$. Let B be the point on CP produced beyond P such that $AB = AC$. Then $ABCD$ is a convex quadrilateral with $AB = AD = CD$, and $\angle DBC = 30^\circ$. In this case A is the circumcenter of $\triangle BCD$.

Example 3. The mistake lies in the argument that if $\angle AEB = \angle ABE$, then $AB = AE$. If $\angle AEB = \angle ABE = 0^\circ$, we cannot conclude that $AB = AE$. Counterexample: suppose ABC is a triangle with $\angle A = 120^\circ$ and $AB \neq AC$.

Example 4. It can easily be verified that $\angle ABP = \angle CBP$ and that $\angle ABQ = \angle CBQ$, so that B, P, Q, and I are collinear. The mistake lies in the argument that if $\angle PBI = \angle QBI$, then $BP : BQ = PI : IQ$. If $\angle PBI = \angle QBI = 0^\circ$, we cannot conclude that $BP : BQ = PI : IQ$.

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