

THE OLYMPIAD CORNER

No. 232

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

As an Olympiad set this issue we give the problems of the 32nd Austrian Mathematics Olympiad, Final Round (Advanced Level), May 30 and 31, 2001. My thanks go to Walther Janous, Ursulinengymnasium, Innsbruck, Austria for translating them and sending them for our use in the *Corner*.

32nd AUSTRIAN MATHEMATICS OLYMPIAD Final Round (Advanced Level) May 30–31, 2001

1. Prove that

$$\frac{1}{25} \sum_{k=0}^{2001} \left\lfloor \frac{2^k}{25} \right\rfloor$$

is a natural number, where $\lfloor x \rfloor$ denotes the greatest whole number less than or equal to x .

2. Determine all triplets of positive real numbers x , y , and z solving the system of equations

$$\begin{aligned} x + y + z &= 6, \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 2 - \frac{4}{xyz}. \end{aligned}$$

3. We are given a triangle ABC having $k(U, r)$ as its circumcircle. Next we construct the 'doubled' circle $k(U, 2r)$ and its two tangents parallel to $c = AB$. Among them we select the one (and designate it c') for which C lies between c and c' . In a similar way we get the tangents a' and b' .

Let $A'B'C'$ be the triangle having its sides on a' , b' , and c' , respectively. Prove: The lines joining the mid-points of corresponding sides of the two triangles intersect in a single point.

4. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$, such that for all real numbers x and y the functional equation $f(f(x)^2 + f(y)) = x \cdot f(x) + y$ is satisfied.

5. Determine all whole numbers m for which all solutions of the equation $3x^3 - 3x^2 + m = 0$ are rational numbers.

6. We are given a semicircle s with diameter AB . On s we choose any two points C and D such that $AC = CD$. The tangent at C intersects line BD in a point E . Line AE intersects s at point F .

Prove that $CD < FD$.

Next we give a set of five Klamkin Quickies. Thanks go to Murray S. Klamkin, University of Alberta, Edmonton, AB for creating them. Try them before looking at the "Quickie Solutions".

FIVE KLAMKIN QUICKIES October 2003

1. If, in the spherical triangle ABC , $a + b + c = \pi$, prove that

$$\cos A + \cos B + \cos C = 1.$$

2. Sum the following two n -term series for $\theta = 30^\circ$:

- (i) $1 + \frac{\cos \theta}{\cos \theta} + \frac{\cos(2\theta)}{\cos^2 \theta} + \frac{\cos(3\theta)}{\cos^3 \theta} + \cdots + \frac{\cos((n-1)\theta)}{\cos^{n-1} \theta}$, and
- (ii) $\cos \theta \cos \theta + \cos^2 \theta \cos(2\theta) + \cos^3 \theta \cos(3\theta) + \cdots + \cos^n \theta \cos(n\theta)$.

3. Let $a, b, c \in \mathbb{R}$, and let $s = (a + b + c)/2$. Determine the maximum and minimum values of

- (i) $\cos(s) \cos(s-a) \cos(s-b) \cos(s-c)$
 $+ \sin(s) \sin(s-a) \sin(s-b) \sin(s-c)$,
- (ii) $\cos(s) \cos(s-a) \cos(s-b) \cos(s-c)$
 $- \sin(s) \sin(s-a) \sin(s-b) \sin(s-c)$.

4. A farmer wanted to fence off a plot of land along a straight river using 300 metres of fencing. He asked his wife, who was mathematically inclined, to give him the dimensions of a rectangle which would maximize the area of the plot (assuming no fencing was to be used along the river). After thinking about it, his wife said he would do better to use a trapezoid which was not a rectangle. Determine the ratio of the trapezoid of maximum area to the rectangle of maximum area.

5. Prove that, for any quadrilateral $ABCD$, the two diagonals AC and BD are orthogonal if and only if

$$AB^2 + CD^2 = BC^2 + DA^2.$$

Here are Murray Klamkin's official "Quickie Solutions" to his puzzles above.

SOLUTIONS TO FIVE KLAMKIN QUICKIES October 2003

1. By the Law of Cosines for spherical triangles

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \text{ etc.}$$

Hence,

$$\sum \cos A = \sum \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

(Sums and products here and subsequently are cyclic.) For this to equal 1, it is sufficient that

$$\sum \sin a \cos a = \sum \sin a \cos b \cos c + \prod \sin a.$$

Since $a + b + c = \pi$, we have the known trigonometric identities

$$\sum \sin 2a = 4 \prod \sin a \text{ and } \sum \sin a \cos b \cos c = \prod \sin a,$$

which means we are done.

Incidentally, we immediately have the dual result that if $A + B + C = 2\pi$, then $\cos a + \cos b + \cos c = -1$.

2. Since $\cos(k \cdot 30^\circ)$ is periodic, one could break up the sums into a bunch of different geometric series, but this is rather tedious. By writing $\cos(k\theta) = \Re e(e^{ik\theta})$, we can sum each series for general θ .

(i) Letting $x = e^{i\theta} / \cos \theta$, the given sum is

$$\begin{aligned} \Re e \left(\sum_{k=0}^{n-1} x^k \right) &= \Re e \left(\frac{1 - x^n}{1 - x} \right) = \Re e \left(\frac{1 - e^{in\theta} / \cos^n \theta}{-i \tan \theta} \right) \\ &= \frac{\sin(n\theta)}{\sin \theta \cos^{n-1} \theta}. \end{aligned}$$

(ii) Here, with $x = e^{i\theta} \cos \theta$, the given sum is

$$\begin{aligned} \Re e \left(\sum_{k=1}^n x^k \right) &= \Re e \left(\frac{x(1 - x^n)}{1 - x} \right) = \Re e \left(\frac{e^{i\theta} \cos \theta (1 - e^{in\theta} \cos^n \theta)}{e^{i\theta} (e^{-i\theta} - \cos \theta)} \right) \\ &= \frac{\sin(n\theta) \cos^{n+1} \theta}{\sin \theta}. \end{aligned}$$

Finally, we let θ be 30° in the above two sums.

3. We start with two identities, which are not very well-known:

$$\begin{aligned} & 4 \cos(s) \cos(s-a) \cos(s-b) \cos(s-c) \\ &= 2 \cos a \cos b \cos c - 1 + \cos^2 a + \cos^2 b + \cos^2 c, \\ & 4 \sin(s) \sin(s-a) \sin(s-b) \sin(s-c) \\ &= 2 \cos a \cos b \cos c + 1 - \cos^2 a - \cos^2 b - \cos^2 c. \end{aligned}$$

Hence, (i) reduces to $\cos a \cos b \cos c$, whose extreme values are ± 1 , and (ii) reduces to $(\cos^2 a + \cos^2 b + \cos^2 c - 1)/2$, whose extreme values are 1 and $-1/2$.

4. By reflecting the two figures across their open sides we end up with a quadrilateral and a hexagon each with perimeter 600. By the isoperimetric theorem, both figures have maximum area when they are regular. For the rectangle, which will be half a square, the dimensions must be width 75, length 150. The trapezoid will be half of a regular hexagon of side 100. Thus, the ratio of the two maximum areas is $7500\sqrt{3}/11250 = 2\sqrt{3}/3 \approx 1.155$.

5. Let vectors from A to B, C, D be given by $\vec{B}, \vec{C}, \vec{D}$, respectively. Then we have the identity

$$\begin{aligned} AB^2 + CD^2 - BC^2 - DA^2 &= \vec{B}^2 + (\vec{C} - \vec{D})^2 - (\vec{B} - \vec{C})^2 - \vec{D}^2 \\ &= 2\vec{C} \cdot (\vec{B} - \vec{D}). \end{aligned}$$

Thus, $AB^2 + CD^2 = BC^2 + DA^2$ if and only if $\vec{C} \cdot (\vec{B} - \vec{D}) = 0$, which is true if and only if AC and BD are orthogonal.

Comment. This result applies to all quadrilaterals, simple or not, planar or not.

As our first set of solutions this issue, we present answers from our readers to problems of the 1997 Iranian Mathematical Olympiad, Second Round, given [2001 : 233–234].

1. Suppose that S is a finite set of real numbers with the property that any two distinct elements of S will form an arithmetic progression with another element of S . Give an example of such a set with 5 elements and prove that no such set exists with at least 6 elements.

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the write-up by Bornsztein.

Suppose that S is a finite set, with at least two elements, and having the property \mathcal{P} : “any two distinct elements of S will form an arithmetic progression with another element of S ”. An example of such a set with 5 elements is $\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$.

If we subtract the same real number from each of the elements of S , we obtain a new set which has the property \mathcal{P} . And, if we multiply each of the elements of S by a non-zero constant, we also obtain a new set which has the property \mathcal{P} . Thus, with no loss of generality, we may suppose that $\min S = 0$ and $\max S = 1$.

From \mathcal{P} , it follows that $\frac{1}{2} \in S$. Define $S' = \{x \in S : \frac{1}{2} < x < 1\}$. Suppose that $S' \neq \emptyset$. We will show that S' cannot contain any number other than $\frac{2}{3}$. For the purpose of contradiction, let us assume this is false. Since $S' \subset S$, the set S' is finite. Thus, we may consider the number $a \in S'$ such that $|\frac{2}{3} - a|$ is minimally positive.

We have $\frac{1}{2} > \frac{a}{2}$ and $\frac{a}{2} \in S$ from the property \mathcal{P} applied to 0 and a (noting that $2a \notin S$ and $-a \notin S$, since $2a > 1$ and $-a < 0$). In the same way, $b = \frac{1+a}{2}$ must belong to S . But $b \in S'$ and $|\frac{2}{3} - b| = \frac{1}{4}|\frac{2}{3} - a|$. This contradicts the minimality property of a . Thus, S' cannot contain more than one element, and the unique number that S' may contain is $\frac{2}{3}$.

We prove in the same manner that the set $S'' = \{x \in S : 0 < x < \frac{1}{2}\}$ cannot contain more than one element, and the unique number that S'' may contain is $\frac{1}{3}$.

It follows that a finite set with the property \mathcal{P} has at most 5 elements.

2. Suppose that ten points are given in the plane such that any five of them contains four points which are concyclic. What is the largest number N for which we can correctly say: "At least N of the ten points lie on a circle"? ($4 \leq N \leq 10$.)

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution by Bornsztein.

The answer is 9.

More generally, let $n \geq 5$ be an integer. Let S be a set of n points in the plane such that among any five of them, four are concyclic. Denote by $f(n)$ the largest integer k for which we can correctly say: "At least k of the n points lie on a circle".

We will prove the following claim.

Claim. $f(6) = 4$, and $f(n) = n - 1$ if $n \geq 5$ and $n \neq 6$.

Lemma. There do not exist seven distinct points M_1, M_2, \dots, M_7 in the plane such that each of the following seven sets is concyclic and the seven circles thus defined are distinct.

- | | |
|--------------------------|--------------------------|
| (1) M_1, M_2, M_3, M_4 | (2) M_1, M_2, M_5, M_6 |
| (3) M_3, M_4, M_5, M_6 | (4) M_2, M_3, M_5, M_7 |
| (5) M_1, M_4, M_5, M_7 | (6) M_1, M_3, M_6, M_7 |
| (7) M_2, M_4, M_6, M_7 | |

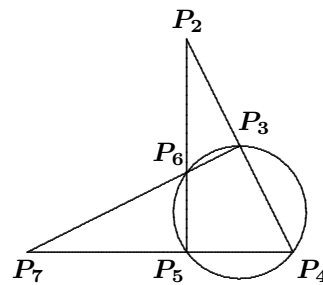
Proof of the Lemma. Suppose, for the purpose of contradiction, that there exist 7 such points. Let f be an inversion with pole M_1 . Let $P_i = f(M_i)$ for $i = 2, \dots, 7$. Then we have

- (1') P_2, P_3, P_4 are collinear.
- (2') P_2, P_5, P_6 are collinear.
- (3') P_3, P_4, P_5, P_6 lie on a circle, say Γ .
- (4') P_2, P_3, P_5, P_7 are concyclic.
- (5') P_4, P_5, P_7 are collinear.
- (6') P_3, P_6, P_7 are collinear.
- (7') P_2, P_4, P_6, P_7 are concyclic.

Case 1. $P_3P_4P_5P_6$ is convex. Then P_2 and P_7 are exterior to Γ .

Subcase (a). P_3 lies between P_2 and P_4 . Then P_6 lies between P_2 and P_5 .

If P_6 lies between P_7 and P_3 , then P_5 lies between P_7 and P_4 (as in the diagram on the right). It follows that P_6 is interior to triangle $P_2P_4P_7$. Thus, (7') cannot be satisfied.

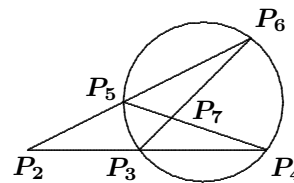


If P_3 lies between P_7 and P_6 , then P_4 lies between P_7 and P_5 . It then follows that P_3 is interior to triangle $P_2P_5P_7$. Thus, (4') cannot be satisfied.

Subcase (b). P_4 lies between P_2 and P_3 . Then P_5 lies between P_2 and P_6 . As above, we prove that either (4') or (7') is not satisfied.

Case 2. $P_3P_4P_6P_5$ is convex. Then P_2 is exterior to Γ , and P_7 is interior to the quadrilateral $P_3P_4P_6P_5$.

Subcase (a). P_3 lies between P_2 and P_4 . Then P_5 lies between P_2 and P_6 (as in the diagram on the right). It follows that P_7 is interior to triangle $P_2P_4P_6$. Thus, (7') cannot be satisfied.

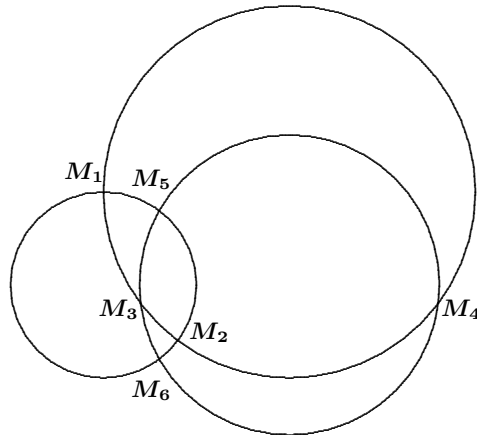


Subcase (b). P_4 lies between P_2 and P_3 . Then P_6 lies between P_2 and P_5 . It follows that P_7 is interior to triangle $P_2P_3P_5$. Thus, (4') cannot be satisfied.

Case 3. $P_3P_5P_4P_6$ is convex. We proceed as in the second case, just interchanging P_2 and P_7 .

In each case, we obtain a contradiction. Thus, the lemma is proved.

Proof of the Claim. First we note that if we choose $n - 1$ points on a circle Γ and another point not on Γ , the set S of these n points satisfies the requirement that among any five points of S , four are concyclic. Thus, $f(n) \leq n - 1$. Moreover, it is clear that $f(n) \geq 4$. Then $f(5) = 4$. The following configuration shows that $f(6) = 4$.



From now on, we suppose that S is a set of $n \geq 7$ points satisfying the requirement in the problem. Suppose (for the purpose of contradiction) that not more than $n - 2$ of the points of S are concyclic.

Let Γ_1 be a circle containing at least four of the n points. With no loss of generality, we may suppose that each of the points M_1, M_2, M_3, M_4 belongs to Γ_1 , and that $M_5 \notin \Gamma_1$, and $M_6 \notin \Gamma_1$.

Given the set $\{M_1, M_2, M_3, M_5, M_6\}$, since $M_5, M_6 \notin \Gamma_1$, then among the four concyclic points in this set, we must have M_5, M_6 , and exactly two of the three other points. With no loss of generality, we may suppose that M_1, M_2, M_5, M_6 lie on a circle, say Γ_2 , where $\Gamma_1 \neq \Gamma_2$.

Given the set $\{M_1, M_3, M_4, M_5, M_6\}$, if M_1 is among the four concyclic points in this set, then without loss of generality M_1, M_i, M_5, M_6 are concyclic for some $i \in \{3, 4\}$. But M_1, M_2, M_5, M_6 are concyclic too, and hence M_1, M_2, M_i, M_5, M_6 are concyclic. Thus, $M_5 \in \Gamma_1$, a contradiction. It follows that M_3, M_4, M_5, M_6 lie on a circle, say Γ_3 , and $\Gamma_3 \neq \Gamma_i$ for $i \in \{1, 2\}$.

Let us suppose (for the purpose of contradiction) that Γ_1 contains at least 5 of the points of S . Then $M_7 \in \Gamma_1$ for some point $M_7 \in S$. Given the set $\{M_1, M_3, M_5, M_6, M_7\}$, we know that four of these points have to be concyclic.

- If M_1, M_3, M_5, M_6 are concyclic, then $M_3 \in \Gamma_2$, and hence $\Gamma_1 = \Gamma_2$, a contradiction.
- If M_1, M_3, M_5, M_7 are concyclic, then $M_5 \in \Gamma_1$, a contradiction.

- If M_1, M_3, M_6, M_7 are concyclic, then $M_6 \in \Gamma_1$, a contradiction.
- If M_1, M_5, M_6, M_7 are concyclic, then $M_7 \in \Gamma_2$ and hence $\Gamma_1 = \Gamma_2$, a contradiction.
- If M_3, M_5, M_6, M_7 are concyclic, then $M_7 \in \Gamma_3$ and hence $\Gamma_1 = \Gamma_3$, a contradiction.

In each case, we obtain a contradiction. It follows that Γ_1 contains exactly 4 points of S .

From the choice of Γ_1 , we deduce that every circle which contains at least 4 points of S , contains exactly 4 points of S . Thus, $M_i \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for $i = 7, \dots, n$.

Let P be any one of the points M_i of S where $i \geq 7$. Given the set $\{M_1, M_2, M_3, M_5, P\}$, since $M_5 \notin \Gamma_1$ and $P \notin \Gamma_1$, the four concyclic points from this set must include both M_5 and P . Similarly, since $P \notin \Gamma_2$, the four concyclic points must also include M_3 .

Case 1. M_2, M_3, M_5, P lie on a circle Γ_4 .

Given the set $\{M_1, M_2, M_4, M_5, P\}$, arguing as above, we see that the points M_4, M_5 , and P must all be included among the four concyclic points. If M_2, M_4, M_5, P are concyclic, then M_2, M_3, M_4, M_5, P are concyclic, which implies that $M_5 \in \Gamma_1$, a contradiction. Therefore, M_1, M_4, M_5, P lie on a circle Γ_5 .

Given the set $\{M_1, M_2, M_3, M_6, P\}$, a similar argument shows that M_1, M_3, M_6, P lie on a circle Γ_6 .

Given the set $\{M_2, M_4, M_5, M_6, P\}$, a similar argument shows that M_2, M_4, M_6, P lie on a circle Γ_7 .

It follows that the seven points $M_1, M_2, M_3, M_4, M_5, M_6, P$ violate the lemma (the seven circles are $\Gamma_1, \Gamma_2, \dots, \Gamma_7$).

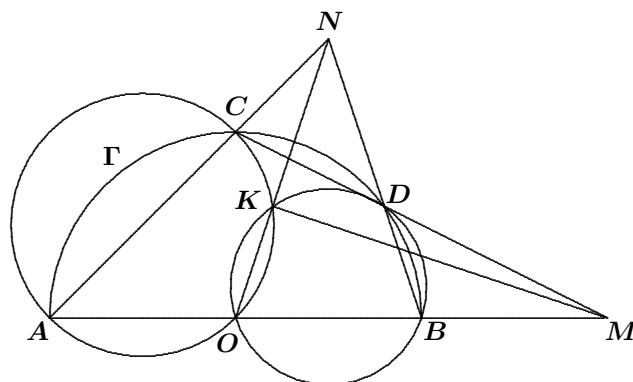
Case 2. M_1, M_3, M_5, P are concyclic.

In the same manner as in case 1 above, by interchanging the roles of M_1 and M_2 , we prove that the seven points $M_2, M_1, M_3, M_4, M_5, M_6, P$ violate the lemma.

It follows that, if $n \geq 7$, then at least $n - 1$ of the points of S are concyclic. That is, if $n \geq 7$, then $f(n) \geq n - 1$. The claim is now proved.

3. Suppose that Γ is a semi-circle with centre O and diameter AB . Assume that M is a point on the extension of AB such that $MA > MB$. A line through M intersects Γ at C and D such that $MC > MD$. Circumcircles of the triangles AOC and BOD will intersect at points O and K . Prove that $OK \perp MK$.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; and D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution of Bataille.



Let N be the point of intersection of AC and BD . The points M and N are conjugate with respect to the circle containing Γ . (We will call this circle Γ as well.) Denote by p, p', p'' the powers of N with respect to the circles $\Gamma, (AOC), (BOD)$, respectively. Then $p = NA \cdot NC = NB \cdot ND$, $p' = NA \cdot NC$, and $p'' = NB \cdot ND$. Thus, $p' = p''$; whence, N lies on the radical axis OK of the circles (AOC) and (BOD) .

Let R be the radius of Γ . From $p = NO^2 - R^2 = NK \cdot NO$, we deduce that $OK \cdot ON = R^2$, which implies that N and K are conjugate with respect to Γ . Since K and M are both conjugates of N , we see that MK is the polar of N with respect to Γ and, as such, is perpendicular to the line $ON (= OK)$ joining N to the centre O of Γ .

4. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{1\}$ such that for all $n \in \mathbb{N} \setminus \{0\}$ we have,

$$f(n+1) + f(n+3) = f(n+5)f(n+7) - 1375.$$

[Ed. The condition $n \in \mathbb{N} \setminus \{0\}$ should actually be $n \in \mathbb{N} \cup \{0\}$, since we need to be able to put $n = 0$ into the above equation in order to have any condition on $f(1)$.]

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the write-up of Evangelopoulos.

Define $a_k = f(2k-1)$ and $b_k = f(2k)$. Then we have, for $k \in \mathbb{N}$,

$$a_k + a_{k+1} = a_{k+2}a_{k+3} - 1375, \quad (1)$$

$$b_k + b_{k+1} = b_{k+2}b_{k+3} - 1375. \quad (2)$$

Replacing k by $k+1$ in (1) and subtracting (1) from the resulting equation, we find that

$$a_{k+2} - a_k = a_{k+3}(a_{k+4} - a_{k+2}).$$

We know that $a_{k+3} \geq 2$. Therefore, $a_{k+2} = a_k$. Otherwise,

$$|a_{k+2} - a_k| > |a_{k+4} - a_{k+2}| > |a_{k+6} - a_{k+4}| > \dots,$$

which is a contradiction. Taking $k = 1$ in (1) we obtain $a_1 + a_2 = a_1 a_2 - 1375$, or $(a_1 - 1)(a_2 - 1) = 1376$. Thus, $a_1 = t + 1$ and $a_2 = \frac{1376}{t} + 1$, where t is any divisor of 1376. Therefore, the sequence satisfies the conditions if and only if

$$a_1 = a_3 = \dots = t + 1, \quad a_2 = a_4 = \dots = \frac{1376}{t} + 1,$$

where t is any divisor of 1376.

Similarly, using (2),

$$b_1 = b_3 = \dots = s + 1, \quad b_2 = b_4 = \dots = \frac{1376}{s} + 1,$$

where s is any divisor of 1376.

Finally, by combining the sequences, the function will be found.

Comment. Bornshtein points out that there are 144 possible functions, since $1376 = 2^5 \times 43$. He also compares this problem with problem #4 of the Vietnamese Mathematical Olympiad 1996, for which a solution was published [2000 : 330–332].

5. Suppose that ABC is an acute triangle with $AC < AB$ and the points O , H , and P are circumcentre, orthocentre, and the foot of the altitude drawn, from C on AB , respectively. The line perpendicular to OP at P intersects the line AC at Q . Prove that $\angle PHQ = \angle BAC$.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; and D.J. Smeenk, Zaltbommel, the Netherlands. We present the comment by Pierre Bornshtein, Pontoise, France.

This problem is #10 of the list proposed to the jury, but not used, at the 37th IMO (1996). A solution was published in **CRUX with MAYHEM** [1998 : 472–473].

6. Suppose that A is a symmetric $(0, 1)$ -matrix such that all of its diagonal entries are 1. Prove that there exist $0 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $A_{i_1} + A_{i_2} + \dots + A_{i_k} = (1, 1, \dots, 1) \pmod{2}$, where A_i is the i^{th} row of A .

Solved by George Evagelopoulos, Athens, Greece.

We prove the above statement by induction on n . For $n = 1$ it is obviously true. Suppose it is true for $n - 1$. We will prove it for n .

Let A be a symmetric $(0, 1)$ -matrix, considered as a matrix over \mathbb{Z}_2 . Define

$$B = \{A_{i_1} + A_{i_2} + \dots + A_{i_k} \mid i_1 < i_2 < \dots < i_k\}.$$

Let $A(i)$ be the matrix obtained by deleting the i^{th} row and the i^{th} column of A . The matrix $A(i)$ is symmetric, and all its diagonal entries are 1. Hence, by the induction hypothesis, $(1, 1, \dots, 1)$ belongs to the column space of $A(i)$. Thus, for $1 \leq i \leq n$, we have either $v(i) = (1, 1, \dots, 0, \dots, 1) \in B$ or $v = (1, 1, \dots, 1) \in B$, where in the former, 0 is in the i^{th} place. If the latter occurs for any i , then we are done; otherwise, for each $1 \leq i \leq n$, we have $v(i) \in B$.

Now, we distinguish two cases:

Case 1. n is even. Then it can be seen easily that $v = \sum_{i=1}^{n/2} v(i)$; whence, $v \in B$, since B is closed under addition.

Case 2. n is odd. Define $w = A_1 + A_2 + \dots + A_n$. Now, since A is symmetric and all the entries on the diagonal are 1, the number of 1's in w is odd and, as a result, the number of 0's is even. Let j_1, j_2, \dots, j_l be the indices corresponding to the 0's in w . Then

$$v = w + \sum_{i=1}^l v(j_i) \in B,$$

and the proof is complete.

The next block of solutions are for the problems of the 1997 Iranian Mathematical Olympiad, Final Round, given [2001 : 234–235].

1. Let n be a positive integer. Prove that there exist polynomials $f(x)$ and $g(x)$ with integer coefficients such that,

$$f(x)(x+1)^{2^n} + g(x)(x^{2^n} + 1) = 2.$$

Solved by George Evagelopoulos, Athens, Greece. Comments by Pierre Borsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB.

Borsztein points out that the problem is equivalent to problem #6 of the list proposed to the jury, but not used, at the 37th IMO (1996). A solution to this was published in [1999 : 135–136].

Klamkin points out that if 2 is replaced by 1 on the right side of the equation, then, since the polynomials multiplying $f(x)$ and $g(x)$ are relatively prime, it is known that integral polynomials exist. Multiplying these by 2 gives the solution.

2. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has the following properties:

(a) $\forall x \in \mathbb{R}, f(x) \leq 1$

(b) $\forall x \in \mathbb{R}, f(x + \frac{13}{42}) + f(x) = f(x + \frac{1}{6}) + f(x + \frac{1}{7})$.

Prove that f is periodic; that is, there exists a non-zero real number T such that for every real number x , we have $f(x + T) = f(x)$.

Solved by George Evagelopoulos, Athens, Greece. Comment by Pierre Borsztein, Pontoise, France.

Borsztein points out that this is problem #7 of the list proposed to the jury, but not used, at the 37th IMO (1996). A solution was published in **CRUX with MAYHEM** [1998 : 466].

3. Suppose that w_1, w_2, \dots, w_k are distinct real numbers with a non-zero sum. Prove that there exist integer numbers n_1, n_2, \dots, n_k such that $\sum_{i=1}^k n_i w_i > 0$ and for any non-identity permutation π on $\{1, 2, \dots, k\}$ we have $\sum_{i=1}^k n_i w_{\pi(i)} < 0$.

Solved by Pierre Borsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution of Evagelopoulos.

First, we prove the following version of the Hardy-Pólya-Littlewood Inequality.

Theorem. Suppose that $a_1 < a_2 < \dots < a_n$ and $b_1 < b_2 < \dots < b_n$ are real numbers. Define

$$\alpha = \min_{1 \leq i < n} (a_{i+1} - a_i) \quad \text{and} \quad \beta = \min_{1 \leq i < n} (b_{i+1} - b_i).$$

Then, for any permutation $\pi \neq 1$, we have

$$\sum_{i=1}^n b_i a_{\pi(i)} \leq \sum_{i=1}^n b_i a_i - \alpha \beta.$$

Proof. Define $A_\pi = \sum_{i=1}^n b_i a_{\pi(i)}$. Let σ be a non-identity permutation with maximum value A_σ . There exist $i < j$ such that $\sigma(i) > \sigma(j)$. Set $\sigma' = \sigma \circ (i j)$. Then

$$A_{\sigma'} = A_\sigma + (a_{\sigma(i)} - a_{\sigma(j)})(b_j - b_i) \geq A_\sigma + \alpha \beta.$$

Thus, $\sigma' = 1$ and the theorem follows.

Without loss of generality, we can assume $w_1 < w_2 < \dots < w_k$. Define $\alpha = \min_{1 \leq i < k} (w_{i+1} - w_i)$ and $s = \left| \sum_{i=1}^k w_i \right| > 0$. Select a natural number $N > \frac{s}{\alpha}$, and set

$$(n_1, n_2, \dots, n_k) = (N, 2N, \dots, kN) + p(1, 1, \dots, 1),$$

where p is the unique integer such that $\sum_{i=1}^k n_i w_i \in (0, s]$.

Now, we have

$$N = \min_{1 \leq i < k} (n_{i+1} - n_i) \quad \text{and} \quad \alpha = \min_{1 \leq i < k} (w_{i+1} - w_i).$$

By the above theorem, for $\pi \neq 1$,

$$\sum_{i=1}^k n_i w_{\pi(i)} \leq \sum_{i=1}^k n_i w_i - N\alpha \leq s - N\alpha < 0.$$

Thus, the proof is complete.

5. Suppose that $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a decreasing continuous function that fulfills the following condition for all $x, y \in \mathbb{R}^+$:

$$f(x+y) + f(f(x) + f(y)) = f(f(x+f(y)) + f(y+f(x))).$$

Prove that $f(x) = f^{-1}(x)$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsstein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution of Bataille.

We first show that $\lim_{x \rightarrow +\infty} f(x) = 0$ and $\lim_{x \rightarrow 0} f(x) = +\infty$.

As f is decreasing and bounded below (by 0), f has a limit $l \geq 0$ as $x \rightarrow +\infty$. Suppose $l > 0$. From the given condition (GC) with $y = x$, we obtain

$$f(2x) + f(2f(x)) = f(2f(x+f(x))).$$

Letting $x \rightarrow +\infty$, the continuity of f yields $l + f(2l) = f(2l)$; whence, $l = 0$, a contradiction. Therefore, $l = 0$.

Similarly, when $x \rightarrow 0$, $f(x)$ tends to either $+\infty$ or a real number $m > 0$. Assume the latter, and consider $g(x) = f(x) - x$. Then g is decreasing and continuous, and $\lim_{x \rightarrow 0} g(x) = m > 0$, $\lim_{x \rightarrow +\infty} g(x) = -\infty$.

Hence, the equation $g(x) = 0$ has a unique solution, say a , in \mathbb{R}^+ . This means that a is the unique fixed point of f . Now, using GC,

$$f(x+a) + f(f(x)+a) = f(f(x+a) + f(a+f(x))).$$

The uniqueness of the fixed point implies that $f(x+a) + f(f(x)+a) = a$.

Letting $x \rightarrow 0$ in this relation, we get $f(a) + f(m+a) = a$, and hence, $f(m+a) = 0$. This is a contradiction, since the range of f is \mathbb{R}^+ . Thus, $\lim_{x \rightarrow 0} f(x) = +\infty$.

It follows that the function f is continuous, decreasing from $(0, +\infty)$ onto $(0, +\infty)$ and, as such, is a bijection.

Lastly, letting $y \rightarrow 0$ in GC, we obtain, for all $x \in (0, +\infty)$,

$$f(x) + 0 = f(0 + f(f(x))), \quad \text{or} \quad f(x) = f(f(f(x))).$$

Since f is bijective, this yields $x = f(f(x))$, and the result $f(x) = f^{-1}(x)$ follows.

We also give the solution of Bornshtein, which provides a different method of attack.

Let f be such a function. Since f is decreasing and continuous, it is a bijection. Thus, we only have to prove that $f(f(x)) = x$ for all $x \in \mathbb{R}^+$.

For all $x \in \mathbb{R}^+$, setting $y = x$, we obtain

$$f(2x) + f(2f(x)) = f(2f(x + f(x))). \quad (1)$$

Replacing x by $f(x)$ in (1), we get

$$f(2f(x)) + f(2f(f(x))) = f(2f(f(x) + f(f(x)))). \quad (2)$$

Subtracting (1) from (2) gives

$$f(2f(f(x))) - f(2x) = f(2f(f(x) + f(f(x)))) - f(2f(x + f(x))). \quad (3)$$

First, suppose for a contradiction that there exists $x \in \mathbb{R}^+$ such that $f(f(x)) > x$. Then, since f is decreasing $f(f(x) + f(f(x))) < f(x + f(x))$. Thus, $f(2f(f(x) + f(f(x)))) > f(2f(x + f(x)))$. It follows that the right side of (3) is positive. But the left side is negative. We have a contradiction.

Similarly, if there exists $x \in \mathbb{R}^+$ such that $f(f(x)) < x$, we can prove that, for any such x , the left side of (3) is positive and the right side of (3) is negative.

We deduce that $f(f(x)) = x$ for all $x \in \mathbb{R}^+$, and we are done.

6. A one story building consists of a finite number of rooms which have been separated by walls. There are one or more doors on some of these walls which can be used to travel in this building. Two of the rooms are marked by S and E . An individual begins walking from S and wants to reach to E .

By a program $P = (P_i)_{i \in I}$ we mean a sequence of R 's and L 's. The individual uses the program as follows: after passing through the n^{th} door, he chooses the rightmost or the leftmost door, meaning that P_n is R or L . For the rooms with one door, any symbol means selecting the door that he had just passed. Notice that the person stops as soon as he reaches E .

Prove that there exists a program P (possibly infinite) with the property that no matter how the structure of the building is, one can reach from S to E by following it. [Editor's comment: one has to assume that there is a way of getting from any room to any other room.]

Solved by George Evagelopoulos, Athens, Greece; and Stan Wagon and Joan Hutchison, Macalester College, St. Paul, MN. We present the write-up by Wagon and Hutchinson.

The problem requires the assumption that the maze-walker, when starting in room S , is facing a definite direction, so that the instructions L and R are unambiguous at the start. Moreover, we require that the rooms not contain “rooms within rooms” that might be inaccessible when restricting to left and right turns. So we assume that each room, once all the doors have been closed, is an empty rectangle (that is, no room can contain a smaller room).

The first step is to prove that any single maze as in the problem can be solved using only left and right turns. For this, identify the maze with a plane embedding of a graph G having two distinguished vertices S and E ; vertices of the graph correspond to rooms, and edges to doors. Further, by discarding extraneous components, assume that G is connected.

Now use a compass bearing to plot a route from S to E : draw a straight line from S to E , perturbing G slightly so that this line passes through no other vertices. The line determines a sequence of faces F_1, \dots, F_n in G so that S lies on F_1 , E lies on F_n , and each F_i is “face-connected” to F_{i+1} (that is, they have a common edge). The initial direction determines a certain face G containing S so that repeated left turns from S will traverse G . Since the faces containing S are face-connected, we may assume by adding in some of these faces that F_1 is the special left-determined face G . Now we may begin the solution by using left turns to start from S and go around face G ($= F_1$) until an edge connecting F_1 and F_2 is traversed. Then switch to right turns, which will start a traversal of F_2 , and continue turning right until an $F_2 | F_3$ edge is reached and traversed. Then switch to lefts. Continue alternating directions until F_n is reached, and conclude by traversing F_n to reach E .

To get a L - R sequence that works on all mazes, enumerate all triples $M_i = (G_i, S_i, E_i)$ as above (the collection of planar graphs is countable because the vertices can be placed on rational points and the edges can be taken to be piecewise linear segments with rational ends). Begin the program with the sequence p_1 that solves M_1 . Follow it with the sequence that solves $(G_2, p_1(S_2), E_2)$. Continue, always appending the sequence that will solve the current graph from the starting point one would be at if one had used the sequence so far formed. This infinite sequence will solve any maze.

I wonder if this idea can be used to design a maze that is solvable using left and right turns only, but is difficult to solve in such a way!

Comment. Jonathan White points out that the problem may have had its origins in the February 1991 Scientific American column of Dewdney on mazes.

A problem from Denmark's Georg Mohr Konkurrencen I Matematik 1996 was generalized by Pierre Bornsztajn [2001 : 240]. He showed that if

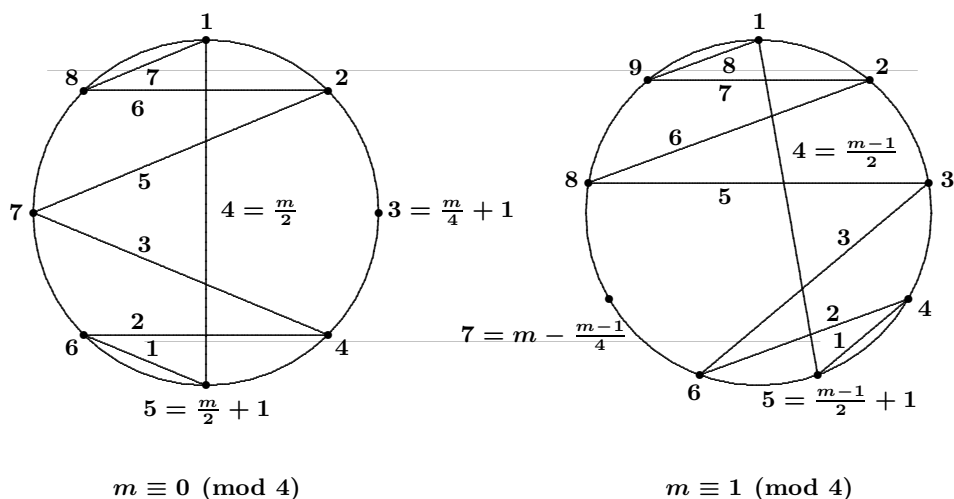
(a) $\pi(k)$ is a permutation of the set $\{1, 2, \dots, n\}$, and

(b) $n \equiv 2$ or $n \equiv 3 \pmod{4}$,

then the numbers $|k - \pi(k)|$ cannot be distinct. The editor asked whether it is possible that $|k - \pi(k)|$ take distinct values in the other cases. Specifically, for $n \equiv 0$ or $n \equiv 1 \pmod{4}$, $n > 0$, can one always find a permutation $\pi : \{1, \dots, n\} \mapsto \{1, \dots, n\}$ for which $|k - \pi(k)|$ takes on all values from 0 to $n - 1$?

Two answers were submitted: one from Pierre Bornsztajn, Pontoise, France; and the other from J. Chris Fisher and Claude Tardif, the University of Regina. We give the answer of Fisher and Tardif.

Here is a picture that indicates one way of defining such a permutation.



That concludes this issue of the *Corner*. Please keep sending me Olympiad contests and your nice solutions.