SKOLIAD No. 72

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are looking for solutions especially from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 April 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item this issue comes from the 2003 Maritime Mathematics Competition written on March 25, 2003. My thanks go out to David Horrocks of the University of Prince Edward Island for forwarding the material to me. We especially invite students in grade 10 (or equivalent) or earlier to send solutions.

2003 Maritime Mathematics Competition
Concours de Mathématiques des Maritimes 2003

1. Si un homme partage un certain nombre de bonbons entre ses enfants, chaque enfant en reçoit quinze et il en reste un. S'il partage le même nombre de bonbons entre ses enfants et deux de leurs amis, chaque enfant en reçoit onze et il en reste trois. De combien de bonbons s'agit-il ?

When a father distributes a number of candies among his children, each child receives 15 candies and there is one left over. If, however, two friends join the group and the candies are redistributed, then each child receives 11 candies and there are three left over. What is the total number of candies?

2. Pour chaque entier strictement positif n, posons

$$f(n) = \left(4(1)^2 - 1\right) \times \left(4(2)^2 - 1\right) \times \cdots \times \left(4n^2 - 1\right).$$

Par example, $f(1) = 3$ et $f(2) = 3 \times 15 = 45$.

Trouver toutes les valeurs de n pour lesquelles $f(n)$ est un carré parfait.

For any positive integer n, define

$$f(n) \;=\; \left(4(1)^2-1\right)\times\left(4(2)^2-1\right)\times\cdots\times\left(4n^2-1\right).$$

For example, $f(1)=3$ and $f(2)=3\times 15=45$.

Find all values of n for which $f(n)$ is a perfect square.

3. Une échelle longue de dix mètres est placée contre un mur vertical. Si le point milieu de l'échelle est deux fois plus distant du sol que du mur, à quelle hauteur l'échelle s'appuie-t-elle contre le mur ?

. . . .

A 10-metre ladder rests against a vertical wall. The mid-point of the ladder is twice as far from the ground as it is from the wall. What height on the wall does the ladder reach?

4. Trouver un nombre à six chiffres dont le premier chiffre est 1 et qui devient trois fois plus grand si le premier chiffre est déplacé à l'autre bout pour devenir le chiffre des unités.

. . . .

Find a 6-digit number with first (that is, leftmost) digit 1 such that if the first digit is transferred to the right, then the number so obtained is three times the original number.

5. Évaluer

$$\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}\,.$$

. . . .

Evaluate

$$\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}\,.$$

6. Trouver toutes les paires d'entiers positifs (x,y) telles que

$$x^2-11y! \;=\; 2003\,.$$

(Par définition, $1!=1$, $2!=1\cdot 2=2$, $3!=1\cdot 2\cdot 3=6$, etc.)

. . . .

Find all pairs of positive integers (x,y) such that

$$x^2-11y! \;=\; 2003\,.$$

(Note that $1!=1$, $2!=(1)(2)=2$, $3!=(1)(2)(3)=6$, etc.)

Next we present the solutions to the 2002 Manitoba Mathematical Contest that appeared in [2003 : 3].

MANITOBA MATHEMATICAL CONTEST, 2002

For students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 20, 2002

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

1. (a) (∗) Solve the equation $x^4 - 3x^2 + 2 = 0$.

(b) (∗) Solve the equation $\dfrac{4}{(x-3)^2} - \dfrac{4}{(x-3)} + 1 = 0$.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

(a)

$$\begin{aligned} x^4 - 3x^2 + 2 &= 0\,, \\ (x^2-1)(x^2-2) &= 0\,. \end{aligned}$$

Thus, $x^2 - 1 = 0$ or $x^2 - 2 = 0$, which means that $x = \pm 1$ or $x = \pm\sqrt{2}$.

(b) From the equation

$$\frac{4}{(x-3)^2} - \frac{4}{x-3} + 1 \;=\; 0\,,$$

we know that $x \neq 3$. Multiplying both sides by $(x-3)^2$, we obtain

$$\begin{aligned} 4 - 4(x-3) + (x-3)^2 &= 0\,, \\ 4 - 4x + 12 + x^2 - 6x + 9 &= 0\,, \\ x^2 - 10x + 25 &= 0\,, \\ (x-5)^2 &= 0\,. \end{aligned}$$

Thus, $x = 5$ is the only root.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

2. (a) (∗) Solve the equation $9x^3 - 9x^2 - 4x + 4 = 0$.

(b) (∗) Thirty-six students took a final exam. The average score of those who passed was 60, the average score of those who failed was 42 and the average of all the scores was 53. How many students did not pass the exam?

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(a) The equation factors to

$$\begin{aligned} (x-1)(9x^2-4) &= 0\,, \\ (x-1)(3x+2)(3x-2) &= 0\,. \end{aligned}$$

Thus, the solutions are $x = 1$, $\pm\frac{2}{3}$.

(b) Let p and f be the number of students who passed and failed the exam, respectively. Then

$$p + f = 36\,, \qquad \text{or} \qquad p = 36 - f\,,$$

and

$$\frac{60p + 42f}{36} = 53\,, \qquad \text{or} \qquad 10p + 7f = 318\,.$$

Substituting the first equation into the second yields

$$\begin{aligned} 10(36 - f) + 7f &= 318\,, \\ 3f &= 42\,, \end{aligned}$$

or $f = 14$. Therefore, 14 students failed the exam.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

3. (a) $(*)$ The area of a rectangle is 3 and its perimeter is 7. What is the length of the diagonal of this rectangle?

(b) $(*)$ In this problem O is the origin, A is the point $(3, 4)$ and B is a point in the first quadrant on the line joining O and A. If the length of AB is 6 what are the coordinates of B?

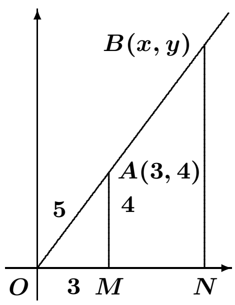
(a) *Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.*

Let l and w represent the length and width of the rectangle. We know that $lw = 3$ and $2(l + w) = 7$, or $l + w = \frac{7}{2}$. Squaring the second equation, the result is $l^2 + w^2 + 2lw = \frac{49}{4}$. Subtracting twice the first equation from this gives $l^2 + w^2 = \frac{49}{4} - 6 = \frac{25}{4}$. Thus, by the Pythagorean Theorem, the length of the diagonal is $\sqrt{\frac{25}{4}} = \frac{5}{2}$.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(b) *Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.*

Triangle AMO is a right-angled 3-4-5 triangle. Also, triangles AMO and BNO are similar, since they are right-angled and share an angle at O. If we let (x, y) be the coordinates of B, we get



$$\begin{aligned} \frac{OB}{OA} &= \frac{NO}{MO} = \frac{BN}{AM}\,, \\ \frac{11}{5} &= \frac{x}{3} = \frac{y}{4}\,. \end{aligned}$$

Thus, $x = \frac{33}{5}$ and $y = \frac{44}{5}$, and $B = \left(\frac{33}{5}, \frac{44}{5}\right)$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

4. (a) ($*$) Solve the equation $\sqrt{3-x}+\sqrt{12-4x}=\sqrt{x-1}$.

(b) ($*$) If p, q and r are the three roots of the equation $x^3-7x^2+3x+1=0$, find the value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$.

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(a)
$$\begin{aligned}\sqrt{3-x}+\sqrt{12-4x} &= \sqrt{x-1}\,,\\ \sqrt{3-x}+2\sqrt{3-x} &= \sqrt{x-1}\,,\\ 3\sqrt{3-x} &= \sqrt{x-1}\,,\\ 9(3-x) &= x-1\,,\\ 27-9x &= x-1\,,\\ x &= \frac{14}{5}\,.\end{aligned}$$

Substituting this value into the original equation verifies that it is a solution.

(b) We have
$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \;=\; \frac{pq+pr+pq}{pqr}\,.$$

For any polynomial equation of the form $ax^3+bx^2+cx+d=0$, where p, q, and r are the roots, we have $qr+pr+pq=c/a$ and $pqr=-d/a$. Thus, for $x^3-7x^2+3x+1=0$, we have

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \;=\; \frac{pq+pr+pq}{pqr} \;=\; \frac{c/a}{-d/a} \;=\; \frac{3}{-1} \;=\; -3\,.$$

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

5. (a) ($*$) If a and b are real numbers such that $\sqrt{a}-\sqrt{b}=\sqrt{2}$ and $a-b=10$, find a and b.

(b) ($*$) If k is a real number such that $3\left(2^{k+3}\right)-2^{2k}=128$, what are the possible values of k?

(a) *Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.*

Squaring the first equation yields $a+b-2\sqrt{ab}=2$. The second equation is equivalent to $a=10+b$. Putting these two together gives

$$\begin{aligned}10+b+b-2\sqrt{b(10+b)} &= 2\,,\\ 2b+8 &= 2\sqrt{b(10+b)}\,,\\ b+4 &= \sqrt{b(10+b)}\,.\end{aligned}$$

Squaring both sides gives $b^2+8b+16=b^2+10b$, which simplifies to $b=8$. This implies that $a=18$. Substituting into the original equations verifies that $(a,b)=(18,8)$ is a solution.

(b) *Official Solution.*
From the original equation, we get

$$\begin{aligned} 3\left(2^{k+3}\right) - 2^{2k} &= 128\,, \\ 3\left(2^{k}\right)\left(2^{3}\right) - \left(2^{k}\right)^{2} &= 128\,, \\ \left(2^{k}\right)^{2} - 24\left(2^{k}\right) + 128 &= 0\,, \\ \left(2^{k} - 8\right)\left(2^{k} - 16\right) &= 0\,. \end{aligned}$$

It follows that $2^k = 8$ or $2^k = 16$, and $k = 3$ or $k = 4$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON; and Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

6. (a) In triangle ABC, $\angle BAC = 60^\circ$, $\angle ACB = 90^\circ$, and D is on BC. If AD bisects $\angle BAC$, prove that $DB = 2CD$.

(b) In triangle ABC, $AC = BC = 5$, and $AB = 8$. What is the radius of the circle which passes through A, B, and C?

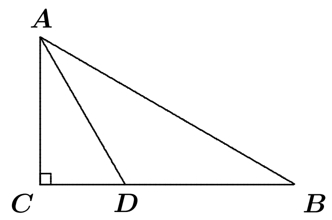
Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

(a) In $\triangle BAC$ we have

$$AC = AB \cos 60^\circ\,.$$

That is, $\dfrac{AB}{AC} = 2$. By the Angle Bisector Theorem,

$$\frac{DB}{DC} = \frac{AB}{AC} = 2\,.$$



Therefore, $DB = 2CD$.

(b) Drop a perpendicular from C to meet AB at D. By symmetry, D is the mid-point of AB. Thus, $AD = DB = 4$. Also by symmetry, $\angle ADC = \angle BDC = 90^\circ$. Thus, since $\triangle CDA$ is right-angled, $\sin A = \frac{3}{5}$. By the Sine Rule,

$$\frac{a}{\sin A} = 2R\,,$$

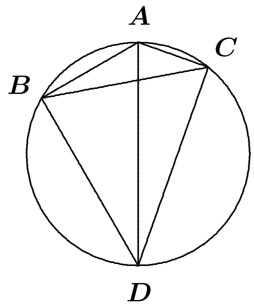
where R is the radius we seek and $a = BC = 5$. Then

$$R = \frac{5}{2\left(\frac{3}{5}\right)} = \frac{25}{6}\,.$$

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

7. A, B, and C are points on a circle of radius 3. In triangle ABC, $\angle ACB = 30^\circ$ and $AC = 2$. Find BC.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.



Let D be the point on the circle such that AD is a diameter. The length AD is then 6. Since $\angle ADB$ subtends the same arc as $\angle ACB$, we have $\angle ADB = 30^\circ$. Since AD is a diameter, we have $\angle ABD = 90^\circ$. Then $\angle BAD = 60^\circ$. Therefore,

$$\begin{aligned} AB : BD : AD &= 1 : \sqrt{3} : 2 \\ &= 3 : 3\sqrt{3} : 6 \,. \end{aligned}$$

Thus, $AB = 3$ and $BD = 3\sqrt{3}$.

Now, $\angle ACD = 90^\circ$, since AD is a diameter. Using the Pythagorean Theorem in $\triangle ADC$ yields

$$CD^2 \;=\; 6^2 - 2^2 \;=\; 32 \,;$$

that is, $CD = \sqrt{32} = 4\sqrt{2}$. Thus, by Ptolemy's Theorem, we have

$$\begin{aligned} AB \cdot CD + BD \cdot AC &= AD \cdot BC \,, \\ 12\sqrt{2} + 6\sqrt{3} &= 6 \cdot BC \,, \\ BC &= 2\sqrt{2} + \sqrt{3} \,. \end{aligned}$$

Also solved by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON; and Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

8. If x and y are real numbers, prove that $x^3y + xy^3 \le x^4 + y^4$.

Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

Without loss of generality, assume that $x \le y$; then, $x^3 \le y^3$. Let $a_n \in \{x^3, y^3\}$ and $b_n \in \{x, y\}$ for $n \in \{1, 2\}$. By the Rearrangement Inequality, $S = a_1b_1 + a_2b_2$ is maximized when the sequences are similarly ordered, and minimized when the sequences are oppositely ordered. In our case,

$$(x^3)(y) + (y^3)(x) \;\le\; S \;\le\; (x^3)(x) + (y^3)(y) \,.$$

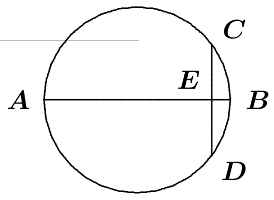
Therefore, $x^3y + xy^3 \le x^4 + y^4$, for all x, $y \in \mathbb{R}$.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

9. A, B, C, and D are points on a circle. AB is the diameter. CD is perpendicular to AB and meets AB at E. If AB and CD are integers and $AE - EB = \sqrt{7}$, find AE.

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

Let diameter $AB = d$ and chord $CD = c$, where c, $d \in \mathbb{Z}$. Since chord CD is perpendicular to diameter AB, we know that $CE = ED = c/2$. Since $AE + EB = d$ and $AE - EB = \sqrt{7}$, we get $2EB + \sqrt{7} = d$. Thus, $EB = \dfrac{d - \sqrt{7}}{2}$. Also, $AE = EB + \sqrt{7} = \dfrac{d + \sqrt{7}}{2}$.



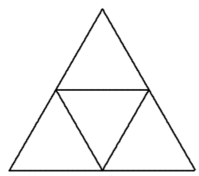
The Intersecting Chord Theorem tells us $(CE)(ED) = (AE)(EB)$; that is, $c^2 = d^2 - 7$. Since c and d are integers, c^2 and d^2 are perfect squares with a difference of 7. The only solution is $c = 3$ and $d = 4$, which gives $AE = \dfrac{4 + \sqrt{7}}{2}$.

Also solved by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

10. Nine points, no three of which lie on the same straight line, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

Divide the given triangle into four smaller equilateral triangles of side length 2, as in the diagram. Each of the smaller equilateral triangles has area $\frac{1}{2}(2)(2)\sin 60^\circ = \sqrt{3}$. By the Pigeonhole Principle, at least $\lceil \frac{9}{4} \rceil = 3$ points must be in one of the smaller triangles. These 3 points then form a triangle whose area is no more than $\sqrt{3}$.



Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

That brings us to the end of another issue of Skoliad. This month a copy of **MATHEMATICAL MAYHEM Vol. 1** goes to Jennifer Park. Congratulations Jennifer! Continue sending in contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **~~Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans~~ Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

`mayhem-editors@cms.math.ca`

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, 2191 Saturn Crescent, Orleans, ON K4A 3T6, ou par courriel à

`mayhem-editors@cms.math.ca`

N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *premier avril 2004*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M107. *Proposé par l'Équipe de Mayhem.*

Soit a et b les longueurs des côtés de l'angle droit d'un triangle rectangle. Un cercle de rayon r touche les côtés et a son centre situé sur l'hypoténuse. Montrer que

$$\frac{1}{a}+\frac{1}{b} = \frac{1}{r}.$$

A right-angled triangle has legs of length a and b. A circle of radius r touches the two legs and has its centre on the hypotenuse. Show that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{r}.$$

M108. *Proposé par l'Équipe de Mayhem.*

Dans un cube dont on a coupé les huit sommets par des plans, combien de diagonales joignant les 24 nouveaux 'sommets' sont-elles comprises entièrement dans le cube ?

. . . .

Given a cube with its eight corners cut off by planes, how many diagonals joining the 24 new 'corners' lie completely inside the cube?

M109. *Proposé par l'Équipe de Mayhem.*

Si tous les plifs sont des plofs et si certains plafs sont des plifs, lesquel des énoncés X, Y, Z doivent être vrais ?

X : Tous les plifs sont des plafs.

Y : Certains plofs sont des plafs.

Z : Certains plifs ne sont pas des plafs.

. . . .

If all plinks are plonks and some plunks are plinks, which of the statements X, Y, Z must be true?

X: All plinks are plunks.

Y: Some plonks are plunks.

Z: Some plinks are not plunks.

M110. *Proposé par l'Équipe de Mayhem.*

A partir d'un nombre distinct de 1, on construit un nouveau nombre en divisant le nombre de départ augmenté d'une unité par le nombre de départ diminué d'une unité. On recommence le processus avec le nouveau nombre. Qu'arrive-t-il ? Expliquez !

. . . .

Given any starting number (other than 1), get a new number by dividing the number 1 larger than your starting number by the number 1 smaller than your starting number. Then do the same with this new number. What happens? Explain!

M111. *Proposé par l'Équipe de Mayhem.*

Un nombres-croisés est comme un mots-croisés, sauf que les réponses sont des nombres, un chiffre par case. Quelle est la somme de tous les chiffres dans la solution de ce nombres-croisés ?

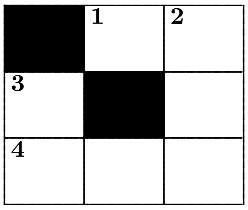
DEFINITIONS

Horizontal

1. Voir 3 Vertical
3. Cube
4. Cinq fois 3 Vertical

Vertical

2. Carré
3. Quatre fois 1 Horizontal



A crossnumber is like a crossword except that the answers are numbers with one digit in each square. What is the sum of all the digits in the solution to this crossnumber?

CLUES

Across

1. See 3 Down
3. A cube
4. Five times 3 Down

Down

2. A Square
3. Four times 1 Across

M112. *Proposé par l'Équipe de Mayhem.*

Déterminer le quotient de l'aire totale de l'hexagone régulier $ABCDEF$ et de l'aire du triangle GDE, si G est le point milieu de AB.

. . . .

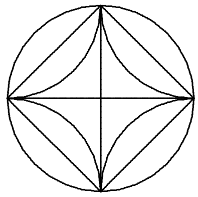
Given that $ABCDEF$ is a regular hexagon and G is the mid-point of AB, determine the ratio of the total area of hexagon $ABCDEF$ to the area of triangle GDE.

Mayhem Solutions

M57. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Quatre points sont egalement espacés autour d'un cercle ayant un rayon r. Le cercle est donc divisé par 4 arcs égaux. Renversez les arcs en laissant le point du bout en place. Trouvez l'aire de la figure ainsi obtenue.

Solution de Robert Bilinski, Outremont, QC.



Puisque les quatre points sur le cercle sont également espacés, le quadrilatère formé par les quatre points est un carré. On remarque que la différence entre les aires du cercle et du carré est la même qu'entre le carré et l'étoile formée par le renversement des arcs de cercle. Le cercle a pour aire πr^2.

Le carré est formé de quatre triangles isocèles rectangles de côtés égaux r et d'hypoténuse $\sqrt{2}r$. Puisque l'hypoténuse des triangles est le côté du carré, son aire est $\left(\sqrt{2}r\right)^2$, soit $2r^2$. La différence entre l'aire du cercle et l'aire du carré est $(\pi - 2)r^2$. Donc l'aire de l'étoile est $(4 - \pi)r^2$.

M58. *Proposed by the Mayhem Staff.*

Find all positive integers x and y which satisfy the equation $x^y = y^x$.

Solution by Mihály Bencze, Brasov, Romania.

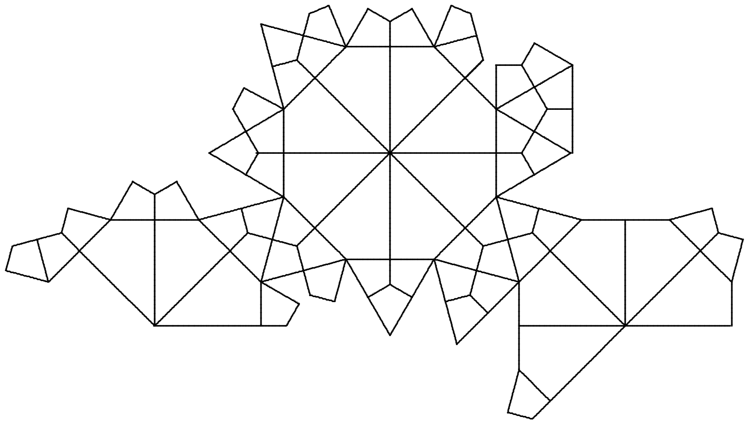
The equation is trivially true if $x = y$. We will search for solutions where $x \neq y$.

The original equation implies $\frac{\ln x}{x} = \frac{\ln y}{y}$. If we let $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1 - \ln x}{x^2}$. Therefore, $f(x)$ is increasing on the interval $(0, e)$ and decreasing on $(e, +\infty)$. Hence, if x, $y \in (0, e)$ and $x > y$, then $f(x) > f(y)$; similarly, if x, $y \in (e, +\infty)$ and $x > y$, then $f(x) < f(y)$. Thus, if $x > y$, we must have $y \in (0, e)$ and $x \in (e, +\infty)$. Checking $y = 1$ and $y = 2$ (the only possible values of y), we find that $(x, y) = (4, 2)$ is a solution with $x \neq y$. Therefore, all possible solutions are:

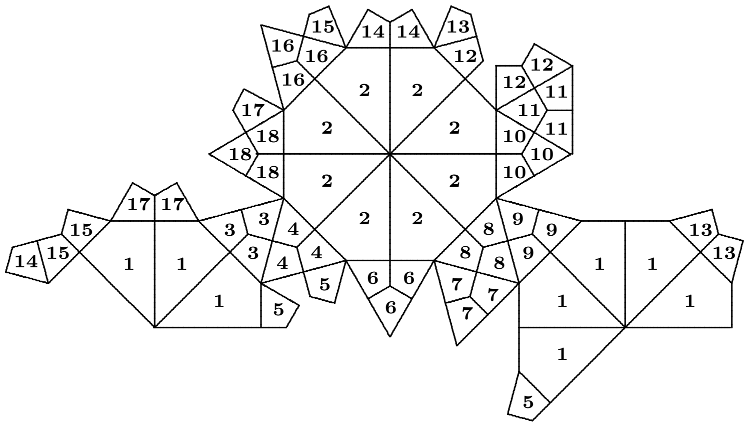
1. $x = y$;
2. $x = 2$, $y = 4$;
3. $x = 4$, $y = 2$.

M59. *Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.*

The diagram below represents the net of a polyhedron in which the faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



Solution by Robert Bilinski, Outremont, QC.



M60. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all positive integers for which $\left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor = n$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution by the proposer.

Let $S_n = \left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor$. We note that $S_1 = 1$, $S_2 = 2$, $S_3 = 4$. For $n > 3$, $S_n > S_{n-1} + 2$. Thus, $S_n = n$ only for $n = 1, 2$.

M61. *Proposed by theMayhem Staff.*

You are given 54 weights which weigh 1^2, 2^2, 3^2, $\ldots$, 54^2. Group these into three sets of equal weight.

Solution by Geneviève Lalonde, Massey, ON.

Note that summing 9 consecutive squares n^2, $(n+1)^2$, $\ldots$, $(n+8)^2$ yields $9n^2+72n+204 = 3(3n^2+24n+68)$. Among these nine squares, we cannot make 3 sets each totalling $3n^2+24n+68$ because $(n+8)^2$ cannot be grouped with two of the other squares to give the desired total (as can be easily checked). If we take

$$\text{Set } 1 \left\{ \begin{array}{c} (n+1)^2 \\ (n+3)^2 \\ (n+8)^2 \end{array} \right. , \qquad \text{Set } 2 \left\{ \begin{array}{c} n^2 \\ (n+5)^2 \\ (n+7)^2 \end{array} \right. , \qquad \text{Set } 3 \left\{ \begin{array}{c} (n+2)^2 \\ (n+4)^2 \\ (n+6)^2 \end{array} \right. ,$$

the first two sets each total $3n^2+24n+74$ and the last totals $3n^2+24n+56$.

Therefore, we can break our 54 weights into 6 groups of 9 and use our sets above within each group of 9, making sure that each of our 3 sets contains two of the subsets that total only $3n^2+24n+56$. There are many solutions, one of which is:

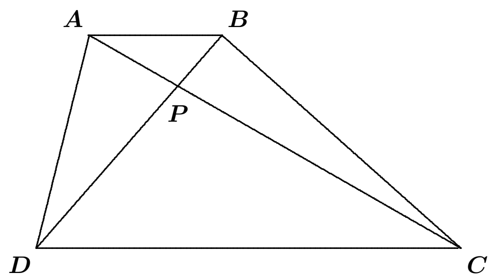
$$\begin{array}{ll} \text{Set 1:} & \{1^2, 6^2, 8^2, 11^2, 13^2, 18^2, 20^2, 22^2, 27^2, 29^2, \\ & \quad 31^2, 36^2, 39^2, 41^2, 43^2, 48^2, 50^2, 52^2\}, \\ \text{Set 2:} & \{2^2, 4^2, 9^2, 10^2, 15^2, 17^2, 21^2, 23^2, 25^2, 30^2, \\ & \quad 32^2, 34^2, 38^2, 40^2, 45^2, 47^2, 49^2, 54^2\}, \\ \text{Set 3:} & \{3^2, 5^2, 7^2, 12^2, 14^2, 16^2, 19^2, 24^2, 26^2, 28^2, \\ & \quad 33^2, 35^2, 37^2, 42^2, 44^2, 46^2, 51^2, 53^2\}. \end{array}$$

Each of these groups sums to $17\,985$.

M62. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Let $ABCD$ be a trapezoid where sides AB and CD are parallel and the diagonals AC and BD intersect at point P. Suppose $AB = 50$, $CD = 160$, and the area of triangle PAD is 2000. Determine the area of the trapezoid.

Solution by Geneviève Lalonde, Massey, ON.



From $AB \parallel CD$, we get $\angle PAB = \angle PCD$ and $\angle PBA = \angle PDC$. Thus, $\triangle PAB$ and $\triangle PCD$ are similar. If we name the heights of P from AB and CD as h_1 and h_2, respectively, we get

$$\frac{h_1}{h_2} = \frac{AB}{CD} = \frac{5}{16}.$$

Then $h_1 = 5h$ and $h_2 = 16h$, for some real number h.

Using the notation $[ABC]$ to represent the area of the figure ABC, we have $[ADC] = \frac{1}{2}(160)(21h) = 1680h$. We also have

$$[ADC] = [ADP] + [PDC] = 2000 + \tfrac{1}{2}(160)(16h) = 2000 + 1280h.$$

Setting these two expressions equal, we get $h = 5$; whence, the height of the trapezoid is $21h = 105$. Therefore, $[ABCD] = \frac{1}{2}(50+160)(105) = 11\,025$.

Pólya's Paragon

Paul Ottaway

For this month's installment, I have decided to explore some of the curious and interesting properties of sequences and series. To begin, I would like to revisit a famous problem that is said to have been solved by the famous mathematician Gauss when he was very young. The story goes that his teacher was frustrated with how quickly he could solve the problems given out in class. Therefore, he was assigned to sum the numbers from 1 to 100, to keep him busy. Remarkably, in almost no time at all, he had solved the problem, much to the teacher's amazement.

Here is the trick he is said to have used:

$$\begin{array}{rcccccc}
S &=& 1 &+& 2 &+ \cdots +& 100\,, \\
S &=& 100 &+& 99 &+ \cdots +& 1\,, \\
2S &=& 101 &+& 101 &+ \cdots +& 101\,, \\
2S &=& 101 \cdot 100\,, &&&& \\
S &=& 5050\,. &&&&
\end{array}$$

By writing the terms forward and backward, we are able to get a very nice expression for twice the sum. The third line is the result of summing the first two lines term by term. Since we know that there are exactly 100 terms, we quickly arrive at the answer.

We would like to be able to use this trick for finding other sums as well. By generalizing, we will now call this a 'technique' which we can use for all sorts of other situations. This time, we will start with an arithmetic sequence where the first term is a, the terms increase by d, and there are n terms. Here is what happens:

$$\begin{aligned}
S &= a + (a+d) + \cdots + \bigl(a+(n-1)d\bigr)\,, \\
S &= \bigl(a+(n-1)d\bigr) + \bigl(a+(n-2)d\bigr) + \cdots + a\,, \\
2S &= \bigl(2a+(n-1)d\bigr) + \bigl(2a+(n-1)d\bigr) + \cdots + \bigl(2a+(n-1)d\bigr)\,, \\
2S &= n\bigl(2a+(n-1)d\bigr)\,, \\
S &= na + \frac{n(n-1)}{2}d\,.
\end{aligned}$$

We can use this formula to determine that the sum of the first n natural numbers is exactly $n(n+1)/2$. To see this, use $a = 1$ and $d = 1$ in the previous equation.

We might now ask ourselves what sort of sums we can achieve when the terms do not form an arithmetic progression. Here are a few more sums with interesting patterns that I will present without proof:

$$\begin{aligned}
\frac{n(n+1)}{2} &= 1+2+\cdots+n\,, \\
\frac{n(n+1)(n+2)}{3} &= 1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)\,, \\
\frac{n(n+1)(n+2)(n+3)}{4} &= 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots \\
&\qquad + n\cdot(n+1)\cdot(n+2)\,.
\end{aligned}$$

We can use these identities to discover even more sums, like the sum of squares shown here:

$$\begin{aligned}
1^2+2^2+\cdots+n^2 &= \bigl(1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)\bigr) \\
&\qquad - (1+2+\cdots+n) \\
&= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{6}\,.
\end{aligned}$$

Finally, I would like to look at numbers called 'triangular' numbers. The k^{th} triangular number is the sum of the first k natural numbers. The first five triangular numbers are 1, 3, 6, 10, and 15. Is there an easy way to find the sum of the first n triangular numbers? The answer is yes! Even though they do not form an arithmetic sequence, we can still find their sum.

$$\begin{aligned}
1+3+6+\cdots+\frac{n(n+1)}{2} &= \frac{1^2+1}{2}+\frac{2^2+2}{2}+\cdots+\frac{n^2+n}{2} \\
&= \frac{1}{2}\left(1^2+2^2+\cdots+n^2\right) \\
&\qquad +\frac{1}{2}\left(1+2+\cdots+n\right) \\
&= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{1}{2}\left(\frac{n(n+1)}{2}\right) \\
&= \frac{n(n+1)(n+2)}{6}.
\end{aligned}$$

Now that you know some useful techniques and identities for finding sums, here are a couple of problems for you to try:

1. Find the sum of the first n cubes. That is, find $1^3+2^3+\cdots+n^3$.

2. Find a relationship between your result from problem 1 and one of the other identities used in this article.

3. Find the sum of the reciprocals of the triangular numbers. That is, find $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\cdots$

   HINT: This is an infinite sum. Start by looking at half this sum, and write each term as the difference of two fractions.

Three-Pile Nim with Blocking

Arthur Holshouser and Harold Reiter

1. Introduction

Nim, also known as Bouton's Nim, is a two-player counter-pickup game that is well known in combinatorial game theory. In this paper we develop a winning strategy for a more complicated variation of Nim, in which exactly one move can be blocked at each stage of the game. Remarkably, the winning strategy for the more complicated version is much simpler than for ordinary (Bouton's) Nim.

Specifically, we explore a three-pile game with two players, a moving player and a blocking player, whose roles alternate between moves. As in ordinary Nim, a move consists of the removal of any number of counters from any single pile. Before each move, including the first move, the blocking player must eliminate exactly one of the moving player's possible moves. For example, if the moving player is confronted with piles of size 6, 10, and 10, the blocking player could forbid the removal of 7 counters from the first of the two 10-counter piles. A forbidden move is forgotten as soon as the next move is made. The winner is the last player to make an allowed move.

2. Bouton's Nim

Before developing the strategy for our game, let us review the strategy for playing Bouton's Nim. The general ideas actually apply to all "last player wins" combinatorial games. The idea is to partition the set $\mathcal{P}$ of all possible positions into two subsets $\mathcal{S}$ and $\mathcal{U}$, where the positions in $\mathcal{S}$ are "safe" to move to and the positions in $\mathcal{U}$ are "unsafe" to move to.

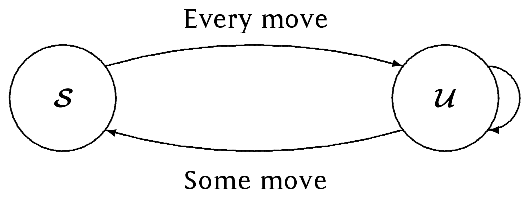
We say that a position v is *accessible* from a position u, and we write $u \longmapsto v$, if there is a move from u to v. Suppose there are two subsets $\mathcal{S}$ and $\mathcal{U}$ which partition $\mathcal{P}$ (that is, $\mathcal{S} \cup \mathcal{U} = \mathcal{P}$ and $\mathcal{S} \cap \mathcal{U} = \phi$) and which possess the following three properties:

(1) From each position v in S, every position accessible from v belongs to $\mathcal{U}$.

(2) From each position u in $\mathcal{U}$, there is at least one position v in $\mathcal{S}$ which is accessible from u.

(3) All terminal positions belong to $\mathcal{S}$.

The sets $\mathcal{S}$ and $\mathcal{U}$ describe a winning strategy. A moving player faced with a position in $\mathcal{U}$ simply moves to a position in $\mathcal{S}$. That player can continue to move to positions in $\mathcal{S}$, ultimately winning the game. Such a strategy is depicted in the diagram below.



In three pile Nim, the members of the set $\mathcal{S}$ can be described as follows. A position in which the piles are of sizes a, b and c is denoted by (a, b, c). Associate with each such position the binary representation of the three integers a, b, c, and align these representations vertically as though we were adding them. If the number of 1's in each column is even, we say that the binary configuration is balanced, and the corresponding position belongs to $\mathcal{S}$. In other words, we take the sum in each column modulo 2. Note that $\mathcal{S}$ and its complement $\mathcal{P} - \mathcal{S}$ satisfy the three properties above.

Consider the game $(13, 15, 17)$.

$$\begin{array}{rcccccc}
13 & = & & 1 & 1 & 0 & 1 \\
15 & = & & 1 & 1 & 1 & 1 \\
17 & = & \boxed{1} & 0 & 0 & \boxed{0} & \boxed{1} \\
\hline
\text{Mod } 2 \text{ Sum} & = & 1 & 0 & 0 & 1 & 1
\end{array}$$

Notice that the first, fourth, and fifth columns have 1's in the bottom row, indicating that these columns have an odd number of 1's. Also note that the three entries in the row with the 17 that are boxed need to be changed so that the columns they occupy become balanced. This can be done by replacing the pile of 17 counters with one having 2 counters. The only winning move is $(13, 15, 17) \longmapsto (13, 15, 2)$. The reason this move is unique is that the 1 in the leftmost column can be eliminated only by a move from the pile with 17 counters. The result can be depicted as:

$$\begin{array}{rccccc}
13 & = & 1 & 1 & 0 & 1 \\
15 & = & 1 & 1 & 1 & 1 \\
2 & = & & & 1 & 0 \\
\hline
\text{Mod } 2 \text{ Sum} & = & 0 & 0 & 0 & 0
\end{array}$$

3. Blocking Nim

Now we consider a game where one move is blocked at each stage. Remarkably, this apparently more complicated game yields a strategy that does not require binary arithmetic. A solution is a partition $(\mathcal{S}, \mathcal{U})$ of the set $\mathcal{P}$ of positions with the following properties:

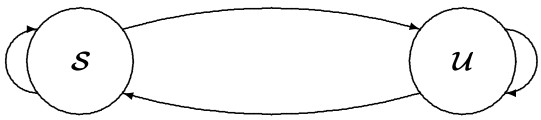
(a) Every terminal position belongs to $\mathcal{S}$.

(b) For each position u in $\mathcal{U}$, there are at least two moves from u to positions in $\mathcal{S}$.

(c) For each position v in $\mathcal{S}$, there is at most one position of $\mathcal{S}$ accessible from v.

The figure below shows how the winning strategy for the blocking game differs from that of the ordinary game.

All moves except one



At least two moves

When denoting a position by (a, b, c), we will generally require $a \leq b \leq c$. In the proof, however, we do not always adhere to this convention because the arithmetic makes it difficult to compare the sizes of the piles once a move has been made. Using this notation, there are two terminal positions: $(0, 0, 0)$ and $(0, 0, 1)$. The position $(0, 0, 1)$ is terminal because the move $(0, 0, 1) \longmapsto (0, 0, 0)$ must be blocked.

Theorem 1 . Let $\mathcal{S}$ denote the set of all positions of the form (a, a, a), where $a \geq 0$, together with the positions (a, b, c) such that $a + b + 1 = c$, and let $\mathcal{U} = \mathcal{P} - \mathcal{S}$. Then the partition $(\mathcal{S},\mathcal{U})$ of $\mathcal{P}$ satisfies conditions (a), (b), and (c) above.

Proof: We can write $\mathcal{S}$ as the union of three sets:

$$\begin{aligned} \mathcal{S} \;\; = \;\; & \{(a,a,a) \mid a \in N\} \cup \{(a,a,c) \mid 2a+1 = c\} \\ & \cup \{(a,b,c) \mid a < b \text{ and } a+b+1 = c\} \,. \end{aligned}$$

We can write $\mathcal{U}$ as:

$$\begin{aligned} \mathcal{U} \;\; = \;\; & \{(a,b,c) \mid a = b < c \text{ and } c \neq 2a+1\} \\ & \cup \{(a,b,c) \mid a < b \leq c \text{ and } a+b+1 \neq c\} \,. \end{aligned}$$

To see condition (a), note that $(0, 0, 0)$ and $(0, 0, 1)$ both belong to $\mathcal{S}$.

Let us show next that property (b) holds. Suppose that (a, b, c) belongs to $\mathcal{U}$. If $a = b < c$ and $c \neq 2a + 1$, then we have two cases to consider: either (i) $2a + 1 > c$ or (ii) $2a + 1 < c$. In case (i), there are two moves to $(c - a - 1, a, c)$, which is a member of $\mathcal{S}$. That is, either of the piles with a counters can be reduced to $c - a - 1$ counters where $c - a - 1 \geq 0$ since $a < c$. In case (ii), there are two moves to positions in $\mathcal{S}$, namely $(a, a, c) \longmapsto (a, a, 2a + 1)$ and $(a, a, c) \longmapsto (a, a, a)$.

On the other hand, if $a < b \leq c$ and $c \neq a+b+1$, we again consider two cases: (i) $a+b+1 > c$ and (ii) $a+b+1 < c$. If $c > a+b+1$, there are two members of $\mathcal{S}$ we could move to, $(a, b, a+b+1)$ and $(a, b-a-1, b)$. The latter position is available because $b - a \geq 1$. In case (ii), the move $(a, b, c) \longmapsto (c-a-1, a, c) \in \mathcal{S}$ is always possible because $0 \leq c-a-1 < b$. Also, the move $(a, b, c) \longmapsto (c-b-1, b, c) \in \mathcal{S}$ is possible when $b < c$ because $0 \leq c-b-1 < a$. When $b = c$, there are always two moves $(a, b, b) \longmapsto (a, b-a-1, b) \in \mathcal{S}$, since $a < b$, and we can reduce either pile with b counters to $b-a-1$.

To prove property (c), let (a, b, c) belong to $\mathcal{S}$. If $a = b = c$, there is no move to another member of $\mathcal{S}$. If the position is of the form (a, a, b) with $2a+1 = b$, there is only one move to another position of $\mathcal{S}$, namely (a, a, a), because any reduction in a pile of size a results in a position (e, a, b) that does not satisfy $e+a+1 = b$. And finally, if (a, b, c) satisfies $a < b$ and $a+b+1 = c$, then there is no move to a position of the form (a, a, a). There is at most one move to a position for which the sum of the first two smaller pile sizes is 1 less than the third. It would involve taking counters from the largest of the three piles. ∎

4. Open Questions

We do not know how to extend this result to games with more than three piles or to games in which the blocking player can block more than one move. There is another version in which instead of blocking a single move, the blocking player is allowed to block a single position. Thus, for example the move from $(2, 2, 2)$ to $(1, 2, 2)$ could be prohibited. We can solve the three-pile game but we cannot extend this result.

5. References

[1] Berlekamp, Conway, and Guy, *Winning Ways*, Academic Press, New York, 1982.

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[3] A. Holshouser, H. Reiter, and J. Rudzinski, *(Last Move) Dynamic One-Pile Nim*, to appear.

[4] A. Holshouser, H. Reiter, and J. Rudzinski, *Dynamic One-Pile Nim*, to appear in The Fibonacci Quarterly.

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THE OLYMPIAD CORNER

No. 232

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

As an Olympiad set this issue we give the problems of the 32^{nd} Austrian Mathematics Olympiad, Final Round (Advanced Level), May 30 and 31, 2001. My thanks go to Walther Janous, Ursulinengymnasium, Innsbruck, Austria for translating them and sending them for our use in the *Corner*.

32^{nd} AUSTRIAN MATHEMATICS OLYMPIAD

Final Round (Advanced Level)

May 30–31, 2001

1. Prove that

$$\frac{1}{25}\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor$$

is a natural number, where $\lfloor x\rfloor$ denotes the greatest whole number less than or equal to x.

2. Determine all triplets of positive real numbers x, y, and z solving the system of equations

$$\begin{aligned} x+y+z &= 6\,, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 2-\frac{4}{xyz}\,. \end{aligned}$$

3. We are given a triangle ABC having $k(U,r)$ as its circumcircle. Next we construct the 'doubled' circle $k(U,2r)$ and its two tangents parallel to $c=AB$. Among them we select the one (and designate it c') for which C lies between c and c'. In a similar way we get the tangents a' and b'.

Let $A'B'C'$ be the triangle having its sides on a', b', and c', respectively. Prove: The lines joining the mid-points of corresponding sides of the two triangles intersect in a single point.

4. Determine all functions $f:\mathbb{R}\mapsto\mathbb{R}$, such that for all real numbers x and y the functional equation $f\big(f(x)^2+f(y)\big)=x\cdot f(x)+y$ is satisfied.

5. Determine all whole numbers m for which all solutions of the equation $3x^3-3x^2+m=0$ are rational numbers.

6. We are given a semicircle s with diameter AB. On s we choose any two points C and D such that $AC = CD$. The tangent at C intersects line BD in a point E. Line AE intersects s at point F.

Prove that $CD < FD$.

Next we give a set of five Klamkin Quickies. Thanks go to Murray S. Klamkin, University of Alberta, Edmonton, AB for creating them. Try them before looking at the "Quickie Solutions".

FIVE KLAMKIN QUICKIES
October 2003

1. If, in the spherical triangle ABC, $a + b + c = \pi$, prove that

$$\cos A + \cos B + \cos C \;=\; 1\,.$$

2. Sum the following two n–term series for $\theta = 30^\circ$:

(i) $1 \;+\; \dfrac{\cos\theta}{\cos\theta} \;+\; \dfrac{\cos(2\theta)}{\cos^2\theta} \;+\; \dfrac{\cos(3\theta)}{\cos^3\theta} \;+\; \cdots \;+\; \dfrac{\cos\big((n-1)\theta\big)}{\cos^{n-1}\theta}$, and

(ii) $\cos\theta\cos\theta \;+\; \cos^2\theta\cos(2\theta) \;+\; \cos^3\theta\cos(3\theta) \;+\; \cdots + \;\cos^n\theta\cos(n\theta)$.

3. Let a, b, $c \in \mathbb{R}$, and let $s = (a+b+c)/2$. Determine the maximum and minimum values of

(i) $\cos(s)\cos(s-a)\cos(s-b)\cos(s-c)$
$\quad + \;\sin(s)\sin(s-a)\sin(s-b)\sin(s-c)$,

(ii) $\cos(s)\cos(s-a)\cos(s-b)\cos(s-c)$
$\quad - \;\sin(s)\sin(s-a)\sin(s-b)\sin(s-c)$.

4. A farmer wanted to fence off a plot of land along a straight river using 300 metres of fencing. He asked his wife, who was mathematically inclined, to give him the dimensions of a rectangle which would maximize the area of the plot (assuming no fencing was to be used along the river). After thinking about it, his wife said he would do better to use a trapezoid which was not a rectangle. Determine the ratio of the trapezoid of maximum area to the rectangle of maximum area.

5. Prove that, for any quadrilateral $ABCD$, the two diagonals AC and BD are orthogonal if and only if

$$AB^2 + CD^2 \;=\; BC^2 + DA^2\,.$$

Here are Murray Klamkin's official "Quickie Solutions" to his puzzles above.

SOLUTIONS TO FIVE KLAMKIN QUICKIES
October 2003

1. By the Law of Cosines for spherical triangles

$$\cos a \;=\; \cos b \cos c + \sin b \sin c \cos A \text{ , etc.}$$

Hence,

$$\sum \cos A \;=\; \sum \frac{\cos a - \cos b \cos c}{\sin b \sin c} \,.$$

(Sums and products here and subsequently are cyclic.) For this to equal 1, it is sufficient that

$$\sum \sin a \cos a \;=\; \sum \sin a \cos b \cos c + \prod \sin a \,.$$

Since $a + b + c = \pi$, we have the known trigonometric identities

$$\sum \sin 2a \;=\; 4 \prod \sin a \text{ and } \sum \sin a \cos b \cos c \;=\; \prod \sin a \,,$$

which means we are done.

Incidentally, we immediately have the dual result that if $A + B + C = 2\pi$, then $\cos a + \cos b + \cos c = -1$.

2. Since $\cos(k \cdot 30^\circ)$ is periodic, one could break up the sums into a bunch of different geometric series, but this is rather tedious. By writing $\cos(k\theta) = \mathfrak{Re}(e^{ik\theta})$, we can sum each series for general θ.

(i) Letting $x = e^{i\theta}/\cos\theta$, the given sum is

$$\begin{aligned} \mathfrak{Re}\left(\sum_{k=0}^{n-1} x^k\right) \;=\; \mathfrak{Re}\left(\frac{1-x^n}{1-x}\right) \;&=\; \mathfrak{Re}\left(\frac{1 - e^{in\theta}/\cos^n\theta}{-i\tan\theta}\right) \\ &=\; \frac{\sin(n\theta)}{\sin\theta\cos^{n-1}\theta} \,. \end{aligned}$$

(ii) Here, with $x = e^{i\theta}\cos\theta$, the given sum is

$$\begin{aligned} \mathfrak{Re}\left(\sum_{k=1}^{n} x^k\right) \;=\; \mathfrak{Re}\left(\frac{x(1-x^n)}{1-x}\right) \;&=\; \mathfrak{Re}\left(\frac{e^{i\theta}\cos\theta(1 - e^{in\theta}\cos^n\theta)}{e^{i\theta}(e^{-i\theta} - \cos\theta)}\right) \\ &=\; \frac{\sin(n\theta)\cos^{n+1}\theta}{\sin\theta} \,. \end{aligned}$$

Finally, we let θ be 30° in the above two sums.

3. We start with two identities, which are not very well-known:

$$\begin{aligned}
&4\cos(s)\cos(s-a)\cos(s-b)\cos(s-c)\\
&\quad= \; 2\cos a\cos b\cos c - 1 + \cos^2 a + \cos^2 b + \cos^2 c\,,\\
&4\sin(s)\sin(s-a)\sin(s-b)\sin(s-c)\\
&\quad= \; 2\cos a\cos b\cos c + 1 - \cos^2 a - \cos^2 b - \cos^2 c\,.
\end{aligned}$$

Hence, (i) reduces to $\cos a\cos b\cos c$, whose extreme values are ± 1, and (ii) reduces to $(\cos^2 a + \cos^2 b + \cos^2 c - 1)/2$, whose extreme values are 1 and $-1/2$.

4. By reflecting the two figures across their open sides we end up with a quadrilateral and a hexagon each with perimeter 600. By the isoperimetric theorem, both figures have maximum area when they are regular. For the rectangle, which will be half a square, the dimensions must be width 75, length 150. The trapezoid will be half of a regular hexagon of side 100. Thus, the ratio of the two maximum areas is $7500\sqrt{3}/11\,250 = 2\sqrt{3}/3 \approx 1.155$.

5. Let vectors from A to B, C, D be given by $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$, respectively. Then we have the identity

$$\begin{aligned}
AB^2 + CD^2 - BC^2 - DA^2 &= \overrightarrow{B}^2 + (\overrightarrow{C} - \overrightarrow{D})^2 - (\overrightarrow{B} - \overrightarrow{C})^2 - \overrightarrow{D}^2\\
&= 2\overrightarrow{C}\cdot(\overrightarrow{B} - \overrightarrow{D})\,.
\end{aligned}$$

Thus, $AB^2 + CD^2 = BC^2 + DA^2$ if and only if $\overrightarrow{C}\cdot(\overrightarrow{B} - \overrightarrow{D}) = 0$, which is true if and only if AC and BD are orthogonal.

Comment. This result applies to all quadrilaterals, simple or not, planar or not.

As our first set of solutions this issue, we present answers from our readers to problems of the 1997 Iranian Mathematical Olympiad, Second Round, given [2001 : 233–234].

1. Suppose that S is a finite set of real numbers with the property that any two distinct elements of S will form an arithmetic progression with another element of S. Give an example of such a set with 5 elements and prove that no such set exists with at least 6 elements.

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the write-up by Bornsztein.

Suppose that S is a finite set, with at least two elements, and having the property $\mathcal{P}$: "any two distinct elements of S will form an arithmetic progression with another element of S". An example of such a set with 5 elements is $\left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$.

If we subtract the same real number from each of the elements of S, we obtain a new set which has the property $\mathcal{P}$. And, if we multiply each of the elements of S by a non-zero constant, we also obtain a new set which has the property $\mathcal{P}$. Thus, with no loss of generality, we may suppose that $\min S = 0$ and $\max S = 1$.

From $\mathcal{P}$, it follows that $\frac{1}{2} \in S$. Define $S' = \left\{x \in S : \frac{1}{2} < x < 1\right\}$. Suppose that $S' \neq \emptyset$. We will show that S' cannot contain any number other than $\frac{2}{3}$. For the purpose of contradiction, let us assume this is false. Since $S' \subset S$, the set S' is finite. Thus, we may consider the number $a \in S'$ such that $\left|\frac{2}{3} - a\right|$ is minimally positive.

We have $\frac{1}{2} > \frac{a}{2}$ and $\frac{a}{2} \in S$ from the property $\mathcal{P}$ applied to 0 and a (noting that $2a \notin S$ and $-a \notin S$, since $2a > 1$ and $-a < 0$). In the same way, $b = \frac{1+\frac{a}{2}}{2}$ must belong to S. But $b \in S'$ and $\left|\frac{2}{3} - b\right| = \frac{1}{4}\left|\frac{2}{3} - a\right|$. This contradicts the minimality property of a. Thus, S' cannot contain more than one element, and the unique number that S' may contain is $\frac{2}{3}$.

We prove in the same manner that the set $S'' = \left\{x \in S : 0 < x < \frac{1}{2}\right\}$ cannot contain more than one element, and the unique number that S'' may contain is $\frac{1}{3}$.

It follows that a finite set with the property $\mathcal{P}$ has at most 5 elements.

2. Suppose that ten points are given in the plane such that any five of them contains four points which are concyclic. What is the largest number N for which we can correctly say: "At least N of the ten points lie on a circle"? $(4 \leq N \leq 10.)$

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution by Bornsztein.

The answer is 9.

More generally, let $n \geq 5$ be an integer. Let S be a set of n points in the plane such that among any five of them, four are concyclic. Denote by $f(n)$ the largest integer k for which we can correctly say: "At least k of the n points lie on a circle".

We will prove the following claim.

Claim. $f(6) = 4$, and $f(n) = n - 1$ if $n \geq 5$ and $n \neq 6$.

Lemma. There do not exist seven distinct points M_1, M_2, $\ldots$, M_7 in the plane such that each of the following seven sets is concyclic and the seven circles thus defined are distinct.

(1)	M_1,	M_2,	M_3,	M_4	(2)	M_1,	M_2,	M_5,	M_6
(3)	M_3,	M_4,	M_5,	M_6	(4)	M_2,	M_3,	M_5,	M_7
(5)	M_1,	M_4,	M_5,	M_7	(6)	M_1,	M_3,	M_6,	M_7
(7)	M_2,	M_4,	M_6,	M_7					

Proof of the Lemma. Suppose, for the purpose of contradiction, that there exist 7 such points. Let f be an inversion with pole M_1. Let $P_i = f(M_i)$ for $i = 2, \ldots, 7$. Then we have

(1′) P_2, P_3, P_4 are collinear.

(2′) P_2, P_5, P_6 are collinear.

(3′) P_3, P_4, P_5, P_6 lie on a circle, say Γ.

(4′) P_2, P_3, P_5, P_7 are concyclic.

(5′) P_4, P_5, P_7 are collinear.

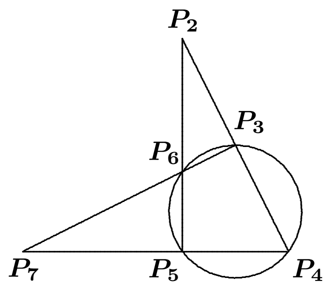
(6′) P_3, P_6, P_7 are collinear.

(7′) P_2, P_4, P_6, P_7 are concyclic.

Case 1. $P_3P_4P_5P_6$ is convex. Then P_2 and P_7 are exterior to Γ.

Subcase (*a*). P_3 lies between P_2 and P_4. Then P_6 lies between P_2 and P_5.

If P_6 lies between P_7 and P_3, then P_5 lies between P_7 and P_4 (as in the diagram on the right). It follows that P_6 is interior to triangle $P_2P_4P_7$. Thus, (7′) cannot be satisfied.

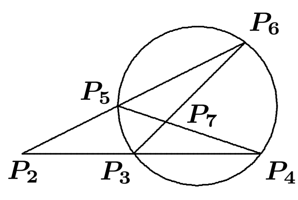


If P_3 lies between P_7 and P_6, then P_4 lies between P_7 and P_5. It then follows that P_3 is interior to triangle $P_2P_5P_7$. Thus, (4′) cannot be satisfied.

Subcase (*b*). P_4 lies between P_2 and P_3. Then P_5 lies between P_2 and P_6. As above, we prove that either (4′) or (7′) is not satisfied.

Case 2. $P_3P_4P_6P_5$ is convex. Then P_2 is exterior to Γ, and P_7 is interior to the quadrilateral $P_3P_4P_6P_5$.

Subcase (*a*). P_3 lies between P_2 and P_4. Then P_5 lies between P_2 and P_6 (as in the diagram on the right). It follows that P_7 is interior to triangle $P_2P_4P_6$. Thus, (7′) cannot be satisfied.

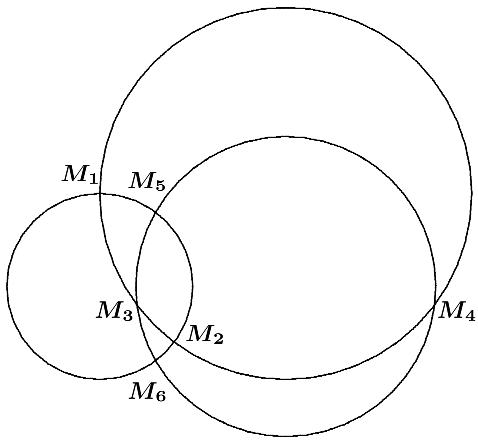


Subcase (*b*). P_4 lies between P_2 and P_3. Then P_6 lies between P_2 and P_5. It follows that P_7 is interior to triangle $P_2P_3P_5$. Thus, (4′) cannot be satisfied.

Case 3. $P_3P_5P_4P_6$ is convex. We proceed as in the second case, just interchanging P_2 and P_7.

In each case, we obtain a contradiction. Thus, the lemma is proved.

Proof of the Claim. First we note that if we choose $n-1$ points on a circle Γ and another point not on Γ, the set S of these n points satisfies the requirement that among any five points of S, four are concyclic. Thus, $f(n) \leq n-1$. Moreover, it is clear that $f(n) \geq 4$. Then $f(5) = 4$. The following configuration shows that $f(6) = 4$.



From now on, we suppose that S is a set of $n \geq 7$ points satisfying the requirement in the problem. Suppose (for the purpose of contradiction) that not more than $n-2$ of the points of S are concyclic.

Let Γ_1 be a circle containing at least four of the n points. With no loss of generality, we may suppose that each of the points M_1, M_2, M_3, M_4 belongs to Γ_1, and that $M_5 \notin \Gamma_1$, and $M_6 \notin \Gamma_1$.

Given the set $\{M_1, M_2, M_3, M_5, M_6\}$, since $M_5, M_6 \notin \Gamma_1$, then among the four concyclic points in this set, we must have M_5, M_6, and exactly two of the three other points. With no loss of generality, we may suppose that M_1, M_2, M_5, M_6 lie on a circle, say Γ_2, where $\Gamma_1 \neq \Gamma_2$.

Given the set $\{M_1, M_3, M_4, M_5, M_6\}$, if M_1 is among the four concyclic points in this set, then without loss of generality M_1, M_i, M_5, M_6 are concyclic for some $i \in \{3, 4\}$. But M_1, M_2, M_5, M_6 are concyclic too, and hence M_1, M_2, M_i, M_5, M_6 are concyclic. Thus, $M_5 \in \Gamma_1$, a contradiction. It follows that M_3, M_4, M_5, M_6 lie on a circle, say Γ_3, and $\Gamma_3 \neq \Gamma_i$ for $i \in \{1, 2\}$.

Let us suppose (for the purpose of contradiction) that Γ_1 contains at least 5 of the points of S. Then $M_7 \in \Gamma_1$ for some point $M_7 \in S$. Given the set $\{M_1, M_3, M_5, M_6, M_7\}$, we know that four of these points have to be concyclic.

- If M_1, M_3, M_5, M_6 are concyclic, then $M_3 \in \Gamma_2$, and hence $\Gamma_1 = \Gamma_2$, a contradiction.
- If M_1, M_3, M_5, M_7 are concyclic, then $M_5 \in \Gamma_1$, a contradiction.

- If M_1, M_3, M_6, M_7 are concyclic, then $M_6 \in \Gamma_1$, a contradiction.
- If M_1, M_5, M_6, M_7 are concyclic, then $M_7 \in \Gamma_2$ and hence $\Gamma_1 = \Gamma_2$, a contradiction.
- If M_3, M_5, M_6, M_7 are concyclic, then $M_7 \in \Gamma_3$ and hence $\Gamma_1 = \Gamma_3$, a contradiction.

In each case, we obtain a contradiction. It follows that Γ_1 contains exactly 4 points of S.

From the choice of Γ_1, we deduce that every circle which contains at least 4 points of S, contains exactly 4 points of S. Thus, $M_i \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for $i = 7, \ldots, n$.

Let P be any one of the points M_i of S where $i \geq 7$. Given the set $\{M_1, M_2, M_3, M_5, P\}$, since $M_5 \notin \Gamma_1$ and $P \notin \Gamma_1$, the four concyclic points from this set must include both M_5 and P. Similarly, since $P \notin \Gamma_2$, the four concyclic points must also include M_3.

Case 1. M_2, M_3, M_5, P lie on a circle Γ_4.

Given the set $\{M_1, M_2, M_4, M_5, P\}$, arguing as above, we see that the points M_4, M_5, and P must all be included among the four concyclic points. If M_2, M_4, M_5, P are concyclic, then M_2, M_3, M_4, M_5, P are concyclic, which implies that $M_5 \in \Gamma_1$, a contradiction. Therefore, M_1, M_4, M_5, P lie on a circle Γ_5.

Given the set $\{M_1, M_2, M_3, M_6, P\}$, a similar argument shows that M_1, M_3, M_6, P lie on a circle Γ_6.

Given the set $\{M_2, M_4, M_5, M_6, P\}$, a similar argument shows that M_2, M_4, M_6, P lie on a circle Γ_7.

It follows that the seven points M_1, M_2, M_3, M_4, M_5, M_6, P violate the lemma (the seven circles are $\Gamma_1, \Gamma_2, \ldots, \Gamma_7$).

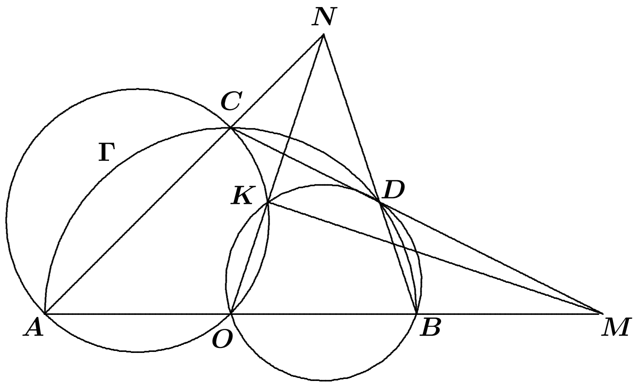
Case 2. M_1, M_3, M_5, P are concyclic.

In the same manner as in case 1 above, by interchanging the roles of M_1 and M_2, we prove that the seven points M_2, M_1, M_3, M_4, M_5, M_6, P violate the lemma.

It follows that, if $n \geq 7$, then at least $n - 1$ of the points of S are concyclic. That is, if $n \geq 7$, then $f(n) \geq n - 1$. The claim is now proved.

3. Suppose that Γ is a semi-circle with centre O and diameter AB. Assume that M is a point on the extension of AB such that $MA > MB$. A line through M intersects Γ at C and D such that $MC > MD$. Circumcircles of the triangles AOC and BOD will intersect at points O and K. Prove that $OK \perp MK$.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; and D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution of Bataille.



Let N be the point of intersection of AC and BD. The points M and N are conjugate with respect to the circle containing Γ. (We will call this circle Γ as well.) Denote by p, p', p'' the powers of N with respect to the circles Γ, (AOC), (BOD), respectively. Then $p = NA \cdot NC = NB \cdot ND$, $p' = NA \cdot NC$, and $p'' = NB \cdot ND$. Thus, $p' = p''$; whence, N lies on the radical axis OK of the circles (AOC) and (BOD).

Let R be the radius of Γ. From $p = NO^2 - R^2 = NK \cdot NO$, we deduce that $OK \cdot ON = R^2$, which implies that N and K are conjugate with respect to Γ. Since K and M are both conjugates of N, we see that MK is the polar of N with respect to Γ and, as such, is perpendicular to the line ON $(= OK)$ joining N to the centre O of Γ.

4. Find all functions $f : \mathbb{N} \to \mathbb{N}\backslash\{1\}$ such that for all $n \in \mathbb{N}\backslash\{0\}$ we have,

$$f(n+1) + f(n+3) \;=\; f(n+5)f(n+7) - 1375\,.$$

[*Ed.* The condition $n \in \mathbb{N}\backslash\{0\}$ should actually be $n \in \mathbb{N} \cup \{0\}$, since we need to be able to put $n = 0$ into the above equation in order to have any condition on $f(1)$.]

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the write-up of Evangelopoulos.

Define $a_k = f(2k-1)$ and $b_k = f(2k)$. Then we have, for $k \in \mathbb{N}$,

$$a_k + a_{k+1} \;=\; a_{k+2}a_{k+3} - 1375\,, \qquad (1)$$

$$b_k + b_{k+1} \;=\; b_{k+2}b_{k+3} - 1375\,. \qquad (2)$$

Replacing k by $k+1$ in (1) and subtracting (1) from the resulting equation, we find that

$$a_{k+2} - a_k \;=\; a_{k+3}(a_{k+4} - a_{k+2})\,.$$

We know that $a_{k+3} \geq 2$. Therefore, $a_{k+2} = a_k$. Otherwise,

$$|a_{k+2} - a_k| \;>\; |a_{k+4} - a_{k+2}| \;>\; |a_{k+6} - a_{k+4}| \;>\; \cdots\,,$$

which is a contradiction. Taking $k = 1$ in (1) we obtain $a_1 + a_2 = a_1 a_2 - 1375$, or $(a_1 - 1)(a_2 - 1) = 1376$. Thus, $a_1 = t + 1$ and $a_2 = \dfrac{1376}{t} + 1$, where t is any divisor of 1376. Therefore, the sequence satisfies the conditions if and only if

$$a_1 \;=\; a_3 \;=\; \cdots \;=\; t + 1, \quad a_2 \;=\; a_4 \;=\; \cdots \;=\; \frac{1376}{t} \;+\; 1\,,$$

where t is any divisor of 1376.

Similarly, using (2),

$$b_1 \;=\; b_3 \;=\; \cdots \;=\; s + 1, \quad b_2 \;=\; b_4 \;=\; \cdots \;=\; \frac{1376}{s} \;+\; 1\,,$$

where s is any divisor of 1376.

Finally, by combining the sequences, the function will be found.

Comment. Bornsztein points out that there are 144 possible functions, since $1376 = 2^5 \times 43$. He also compares this problem with problem #4 of the Vietnamese Mathematical Olympiad 1996, for which a solution was published [2000 : 330–332].

5. Suppose that ABC is an acute triangle with $AC < AB$ and the points O, H, and P are circumcentre, orthocentre, and the foot of the altitude drawn, from C on AB, respectively. The line perpendicular to OP at P intersects the line AC at Q. Prove that $\angle PHQ = \angle BAC$.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; and D.J. Smeenk, Zaltbommel, the Netherlands. We present the comment by Pierre Bornsztein, Pontoise, France.

This problem is #10 of the list proposed to the jury, but not used, at the 37$^{\text{th}}$ IMO (1996). A solution was published in ***CRUX with MAYHEM*** [1998 : 472–473].

6. Suppose that A is a symmetric $(0,1)$–matrix such that all of its diagonal entries are 1. Prove that there exist $0 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $A_{i_1} + A_{i_2} + \cdots + A_{i_k} = (1, 1, \ldots, 1) \pmod 2$, where A_i is the i^{th} row of A.

Solved by George Evagelopoulos, Athens, Greece.

We prove the above statement by induction on n. For $n = 1$ it is obviously true. Suppose it is true for $n - 1$. We will prove it for n.

Let A be a symmetric $(0,1)$–matrix, considered as a matrix over $\mathbb{Z}_2$. Define

$$B \;=\; \{A_{i_1} + A_{i_2} + \cdots + A_{i_k} \mid i_1 < i_2 < \cdots < i_k\}\,.$$

Let $A(i)$ be the matrix obtained by deleting the i^{th} row and the i^{th} column of A. The matrix $A(i)$ is symmetric, and all its diagonal entries are 1. Hence, by the induction hypothesis, $(1, 1, \ldots, 1)$ belongs to the column space of $A(i)$. Thus, for $1 \le i \le n$, we have either $v(i) = (1, 1, \ldots, 0, \ldots, 1) \in B$ or $v = (1, 1, \ldots, 1) \in B$, where in the former, 0 is in the i^{th} place. If the latter occurs for any i, then we are done; otherwise, for each $1 \le i \le n$, we have $v(i) \in B$.

Now, we distinguish two cases:

Case 1. n is even. Then it can be seen easily that $v = \sum_{i=1}^{n/2} v(i)$; whence, $v \in B$, since B is closed under addition.

Case 2. n is odd. Define $w = A_1 + A_2 + \cdots + A_n$. Now, since A is symmetric and all the entries on the diagonal are 1, the number of 1's in w is odd and, as a result, the number of 0's is even. Let $j_1, j_2, \ldots, j_l$ be the indices corresponding to the 0's in w. Then

$$v \;=\; w + \sum_{i=1}^{l} v(j_i) \in B\,,$$

and the proof is complete.

The next block of solutions are for the problems of the 1997 Iranian Mathematical Olympiad, Final Round, given [2001 : 234–235].

1. Let n be a positive integer. Prove that there exist polynomials $f(x)$ and $g(x)$ with integer coefficients such that,

$$f(x)(x+1)^{2^n} + g(x)(x^{2^n}+1) \;=\; 2\,.$$

Solved by George Evagelopoulos, Athens, Greece. Comments by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB.

Bornsztein points out that the problem is equivalent to problem #6 of the list proposed to the jury, but not used, at the 37^{th} IMO (1996). A solution to this was published in [1999 : 135–136].

Klamkin points out that if 2 is replaced by 1 on the right side of the equation, then, since the polynomials multiplying $f(x)$ and $g(x)$ are relatively prime, it is known that integral polynomials exist. Multiplying these by 2 gives the solution.

2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ has the following properties:

(a) $\forall x\in\mathbb{R}$, $f(x)\le 1$

(b) $\forall x\in\mathbb{R}$, $f\left(x+\frac{13}{42}\right)+f(x)=f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right)$.

Prove that f is periodic; that is, there exists a non-zero real number T such that for every real number x, we have $f(x+T)=f(x)$.

Solved by George Evagelopoulos, Athens, Greece. Comment by Pierre Bornsztein, Pontoise, France.

Borsztein points out that this is problem #7 of the list proposed to the jury, but not used, at the 37th IMO (1996). A solution was published in ***CRUX with MAYHEM*** [1998 : 466].

3. Suppose that w_1, w_2, …, w_k are distinct real numbers with a non-zero sum. Prove that there exist integer numbers n_1, n_2, …, n_k such that $\sum_{i=1}^{k} n_i w_i > 0$ and for any non-identity permutation π on $\{1, 2, \ldots, k\}$ we have $\sum_{i=1}^{k} n_i w_{\pi(i)} < 0$.

Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution of Evagelopoulos.

First, we prove the following version of the Hardy-Pólya-Littlewood Inequality.

Theorem. Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ are real numbers. Define

$$\alpha \;=\; \min_{1\le i<n}\,(a_{i+1}-a_i) \qquad\text{and}\qquad \beta \;=\; \min_{1\le i<n}\,(b_{i+1}-b_i)\,.$$

Then, for any permutation $\pi\ne 1$, we have

$$\sum_{i=1}^{n} b_i a_{\pi(i)} \;\le\; \sum_{i=1}^{n} b_i a_i - \alpha\beta\,.$$

Proof. Define $A_\pi = \sum_{i=1}^{n} b_i a_{\pi(i)}$. Let σ be a non-identity permutation with maximum value A_σ. There exist $i<j$ such that $\sigma(i)>\sigma(j)$. Set $\sigma' = \sigma\circ(i\ j)$. Then

$$A_{\sigma'} \;=\; A_\sigma + (a_{\sigma(i)} - a_{\sigma(j)})(b_j - b_i) \;\ge\; A_\sigma + \alpha\beta\,.$$

Thus, $\sigma' = 1$ and the theorem follows.

Without loss of generality, we can assume $w_1 < w_2 < \cdots < w_k$. Define $\alpha = \min\limits_{1\le i<k}(w_{i+1}-w_i)$ and $s = \left|\sum_{i=1}^{k} w_i\right| > 0$. Select a natural number $N > \dfrac{s}{\alpha}$, and set

$$(n_1, n_2, \ldots, n_k) \;=\; (N, 2N, \ldots, kN) + p(1,1,\ldots,1)\,,$$

where p is the unique integer such that $\sum_{i=1}^{k} n_i w_i \in (0, s]$.

Now, we have

$$N \;=\; \min_{1\le i<k}(n_{i+1}-n_i) \qquad \text{and} \qquad \alpha \;=\; \min_{1\le i<k}(w_{i+1}-w_i)\,.$$

By the above theorem, for $\pi \neq 1$,

$$\sum_{i=1}^{k} n_i w_{\pi(i)} \;\le\; \sum_{i=1}^{k} n_i w_i - N\alpha \;\le\; s - N\alpha \;<\; 0\,.$$

Thus, the proof is complete.

5. Suppose that $f:\mathbb{R}^+\to\mathbb{R}^+$ is a decreasing continuous function that fulfills the following condition for all x, $y\in\mathbb{R}^+$:

$$f(x+y)+f\big(f(x)+f(y)\big) \;=\; f\Big(f\big(x+f(y)\big)+f\big(y+f(x)\big)\Big)\,.$$

Prove that $f(x)=f^{-1}(x)$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We give the solution of Bataille.

We first show that $\lim\limits_{x\to+\infty} f(x)=0$ and $\lim\limits_{x\to 0} f(x)=+\infty$.

As f is decreasing and bounded below (by 0), f has a limit $l\ge 0$ as $x\to+\infty$. Suppose $l\ge 0$. From the given condition (GC) with $y=x$, we obtain

$$f(2x)+f\big(2f(x)\big) \;=\; f\Big(2f\big(x+f(x)\big)\Big)\,.$$

Letting $x\to+\infty$, the continuity of f yields $l+f(2l)=f(2l)$; whence, $l=0$, a contradiction. Therefore, $l=0$.

Similarly, when $x\to 0$, $f(x)$ tends to either $+\infty$ or a real number $m>0$. Assume the latter, and consider $g(x)=f(x)-x$. Then g is decreasing and continuous, and $\lim\limits_{x\to 0} g(x)=m>0$, $\lim\limits_{x\to+\infty} g(x)=-\infty$. Hence, the equation $g(x)=0$ has a unique solution, say a, in $\mathbb{R}^+$. This means that a is the unique fixed point of f. Now, using GC,

$$f(x+a)+f\big(f(x)+a\big) \;=\; f\Big(f(x+a)+f\big(a+f(x)\big)\Big)\,.$$

The uniqueness of the fixed point implies that $f(x+a)+f\big(f(x)+a\big)=a$.

Letting $x\to 0$ in this relation, we get $f(a)+f(m+a)=a$, and hence, $f(m+a)=0$. This is a contradiction, since the range of f is $\mathbb{R}^+$. Thus, $\lim\limits_{x\to 0} f(x)=+\infty$.

It follows that the function f is continuous, decreasing from $(0,+\infty)$ onto $(0,+\infty)$ and, as such, is a bijection.

Lastly, letting $y \to 0$ in GC, we obtain, for all $x \in (0, +\infty)$,

$$f(x) + 0 \;=\; f\Big(0 + f\big(f(x)\big)\Big)\,, \qquad \text{or} \qquad f(x) \;=\; f\Big(f\big(f(x)\big)\Big)\,.$$

Since f is bijective, this yields $x = f\big(f(x)\big)$, and the result $f(x) = f^{-1}(x)$ follows.

We also give the solution of Bornsztein, which provides a different method of attack.

Let f be such a function. Since f is decreasing and continuous, it is a bijection. Thus, we only have to prove that $f\big(f(x)\big) = x$ for all $x \in \mathbb{R}^+$.

For all $x \in \mathbb{R}^+$, setting $y = x$, we obtain

$$f(2x) + f\big(2f(x)\big) \;=\; f\Big(2f\big(x + f(x)\big)\Big)\,. \tag{1}$$

Replacing x by $f(x)$ in (1), we get

$$f\big(2f(x)\big) + f\Big(2f\big(f(x)\big)\Big) \;=\; f\Big(2f\big(f(x) + f(f(x))\big)\Big)\,. \tag{2}$$

Subtracting (1) from (2) gives

$$f\Big(2f\big(f(x)\big)\Big) - f(2x) \;=\; f\Big(2f\big(f(x) + f(f(x))\big)\Big) - f\Big(2f\big(x + f(x)\big)\Big)\,. \tag{3}$$

First, suppose for a contradiction that there exists $x \in \mathbb{R}^+$ such that $f\big(f(x)\big) > x$. Then, since f is decreasing $f\Big(f(x) + f\big(f(x)\big)\Big) < f\big(x + f(x)\big)$. Thus, $f\Big(2f\big(f(x) + f(f(x))\big)\Big) > f\Big(2f\big(x + f(x)\big)\Big)$. It follows that the right side of (3) is positive. But the left side is negative. We have a contradiction.

Similarly, if there exists $x \in \mathbb{R}^+$ such that $f\big(f(x)\big) < x$, we can prove that, for any such x, the left side of (3) is positive and the right side of (3) is negative.

We deduce that $f\big(f(x)\big) = x$ for all $x \in \mathbb{R}^+$, and we are done.

6. A one story building consists of a finite number of rooms which have been separated by walls. There are one or more doors on some of these walls which can be used to travel in this building. Two of the rooms are marked by S and E. An individual begins walking from S and wants to reach to E.

By a program $P = (P_i)_{i \in I}$ we mean a sequence of R's and L's. The individual uses the program as follows: after passing through the n^{th} door, he chooses the rightmost or the leftmost door, meaning that P_n is R or L. For the rooms with one door, any symbol means selecting the door that he had just passed. Notice that the person stops as soon as he reaches E.

Prove that there exists a program P (possibly infinite) with the property that no matter how the structure of the building is, one can reach from S to E by following it. [Editor's comment: one has to assume that there is a way of getting from any room to any other room.]

Solved by George Evagelopoulos, Athens, Greece; and Stan Wagon and Joan Hutchison, Macalester College, St. Paul, MN. We present the write-up by Wagon and Hutchinson.

The problem requires the assumption that the maze-walker, when starting in room S, is facing a definite direction, so that the instructions L and R are unambiguous at the start. Moreover, we require that the rooms not contain "rooms within rooms" that might be inaccessible when restricting to left and right turns. So we assume that each room, once all the doors have been closed, is an empty rectangle (that is, no room can contain a smaller room).

The first step is to prove that any single maze as in the problem can be solved using only left and right turns. For this, identify the maze with a plane embedding of a graph G having two distinguished vertices S and E; vertices of the graph correspond to rooms, and edges to doors. Further, by discarding extraneous components, assume that G is connected.

Now use a compass bearing to plot a route from S to E: draw a straight line from S to E, perturbing G slightly so that this line passes through no other vertices. The line determines a sequence of faces $F_1, \ldots, F_n$ in G so that S lies on F_1, E lies on F_n, and each F_i is "face-connected" to F_{i+1} (that is, they have a common edge). The initial direction determines a certain face G containing S so that repeated left turns from S will traverse G. Since the faces containing S are face-connected, we may assume by adding in some of these faces that F_1 is the special left-determined face G. Now we may begin the solution by using left turns to start from S and go around face G $(= F_1)$ until an edge connecting F_1 and F_2 is traversed. Then switch to right turns, which will start a traversal of F_2, and continue turning right until an $F_2 \mid F_3$ edge is reached and traversed. Then switch to lefts. Continue alternating directions until F_n is reached, and conclude by traversing F_n to reach E.

To get a L–R sequence that works on all mazes, enumerate all triples $M_i = (G_i, S_i, E_i)$ as above (the collection of planar graphs is countable because the vertices can be placed on rational points and the edges can be taken to be piecewise linear segments with rational ends). Begin the program with the sequence p_1 that solves M_1. Follow it with the sequence that solves $(G_2, p_1(S_2), E_2)$. Continue, always appending the sequence that will solve the current graph from the starting point one would be at if one had used the sequence so far formed. This infinite sequence will solve any maze.

I wonder if this idea can be used to design a maze that is solvable using left and right turns only, but is difficult to solve in such a way!

Comment. Jonathan White points out that the problem may have had its origins in the February 1991 Scientific American column of Dewdney on mazes.

A problem from Denmark's Georg Mohr Konkurrencen I Matematik 1996 was generalized by Pierre Bornsztein [2001 : 240]. He showed that if

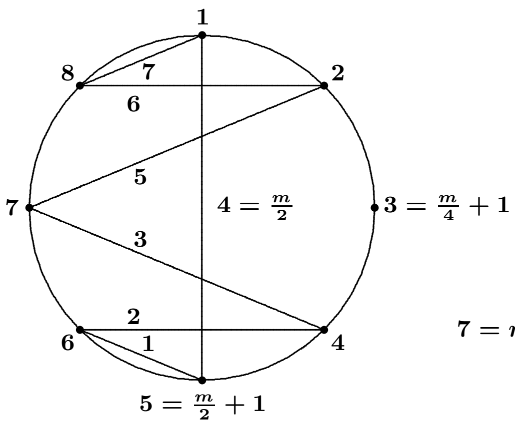
(a) $\pi(k)$ is a permutation of the set $\{1, 2, \ldots, n\}$, and

(b) $n \equiv 2$ or $n \equiv 3 \pmod 4$,

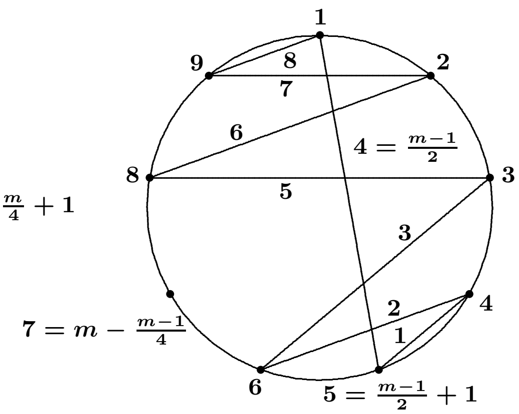
then the numbers $|k - \pi(k)|$ cannot be distinct. The editor asked whether it is possible that $|k - \pi(k)|$ take distinct values in the other cases. Specifically, for $n \equiv 0$ or $n \equiv 1 \pmod 4$, $n > 0$, can one always find a permutation $\pi : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ for which $|k - \pi(k)|$ takes on all values from 0 to $n - 1$?

Two answers were submitted: one from Pierre Bornsztein, Pontoise, France; and the other from J. Chris Fisher and Claude Tardif, the University of Regina. We give the answer of Fisher and Tardif.

Here is a picture that indicates one way of defining such a permuation.



$m \equiv 0 \pmod 4$



$m \equiv 1 \pmod 4$

That concludes this issue of the *Corner*. Please keep sending me Olympiad contests and your nice solutions.

BOOK REVIEWS

John Grant McLoughlin

Probability Games
by Ivan Moscovich, published by Thomas Allen & Son Limited, 2000
ISBN 0-7611-2017-3, hardcover, full colour throughout, 24 pages, CDN\$8.95
Pattern Games
by Ivan Moscovich, published by Thomas Allen & Son Limited, 2001
ISBN 0-7611-2020-3, hardcover, full colour throughout, 24 pages, CDN\$10.95
Reviewed by **Tegan Butler**, *Student, Faculty of Education, University of New Brunswick, Fredericton, New Brunswick.*

With interactive and colourful covers, Ivan Moscovich's *Probability Games* and *Pattern Games* are sure to be hits with persons of all ages. Without even opening *Probability Games*, the reader is faced with an array of kaleidoscopic octagons awaiting to be turned. The task is to match the facing sides of all nine octagons with matching colors. The particular octagon one chooses to begin with determines the time spent achieving the single correct answer. *Pattern Games* provides an equally inviting and intriguing puzzle on the front cover. Also manipulative, this puzzle consists of an 8×8 chessboard with mobile queens. Whether playing with a friend or solo, the idea is to move alternating queens to positions in which each queen is the sole proprietor of that line, be it a row, column or diagonal.

Probability Games kicks off with basic probability concepts, those being permutations, combinations and factorials (all of which the author defines). Further along, the tasks become more perceptual and require more insight. Take the example, "Flip Fraud" (page 10). This takes place at the Hy-Lee Improbable Auction House, where an auctioneer is asking for bids on two charts depicting coin toss outcomes recorded by the one and only 'Thumbs' McDougall. The auctioneer, however, is interrupted by a math teacher who exclaims that one of the charts is a fake! The challenge posed to the reader is to determine why the teacher stated this simply by looking at the two charts.

Pattern Games invites the reader to discover number sequences, identify geometric patterns, decipher codes, and play with optical illusions. Written in a style similar to *Probability Games*, this book also presents the assignments with an air that permits readers to forget that they are even doing mathematics! Consider the puzzle, "Goblins' Getaway" (page 10), in which Fannie has fallen into an underground hideaway inhabited by goblins. In order for Fannie to escape she must pass through all 39 caves without retracing her steps. In a subsequent problem, "Party All Night" (page 11), the toys at Gepetto's Toy Store have played the entire night away. Hurrying to their places on the shelves, they have consequently knocked over their price tags. Arrangements of horses, balls, clowns, horns, and airplanes are illustrated with their total sums provided in all but the final row and column. The reader's assignment is to determine the price of each toy.

Moscovich has organized both books extremely well. Particularly in *Pattern Games*, he has made sure that similar problems (for example, geometric patterns, Pythagorean Theorem) follow a logical sequence. He makes a solid effort and succeeds in relating the problems to everyday encounters in an exciting and enthusiastic way. From Dippity Dan's Ice Cream Parlor in *Probability Games*, in which the reader must figure out the number of three-scoop ice cream possibilities with ten different flavours, to Mrs. Perkins' quilts in *Pattern Games*, where perfect and imperfect squares are linked to quilting, each mathematical proposition exemplifies an incident easily transferable to actual contexts.

Moscovich provides complete solutions that are easily understood and accompanied with appropriate diagrams and illustrations. The delightful illustrations are plentiful throughout and add a nice charm to these wonderful books. Recommended for kids aged eight and above, people of all ages are encouraged to accept Moscovich's invitation to participate in his mathematical challenges.

Mesmerizing Math Puzzles
by Rodolfo Kurchan, published by Sterling Publishing Co., 2000
ISBN 0-8069-3709-2, softcover, 94 pages, CDN$9.95
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

Kurchan's enthusiasm for mathematical puzzling is evident throughout this entertaining collection of 93 problems. The problems are divided into seven chapters: *Numbers*; *Sequences*; *Coded Sums*; *Digits*; *Figures to be Divided*; *Pentominoes*; and *Miscellaneous*. The chapter entitled *Numbers* deals with a range of topics including primes, number properties, divisibility, and logic problems involving natural numbers. For example, one problem arises from the introduction of a novel numerical property, namely, numbers with "exact endings". Numbers such as 512 and 819 have exact endings, since they are each multiples of their final digit. That is, 512 divided by 2 and 819 divided by 9 both leave no remainder. Kurchan identifies 637284591 as a number that uses all digits 1 through 9 once in such an order that each group of three consecutive digits forms a number with an exact ending. He proceeds to offer the challenge: "What is the highest number using nine digits and no repeats that meets this condition?"

In contrast to the opening chapter, *Digits* is almost exclusively devoted to various figures that require proper placement of digits into given spaces to satisfy a range of criteria. *Coded Sums* refer to alphametics where digits must be substituted for letters. *Sequences* features several questions in which the reader is to identify the next term in a sequence. This chapter, in my opinion, is much less interesting than those that surround it. The cover identifies the book as an official Mensa publication. Such publications tend to emphasize such patterning problems. Fortunately the weight of this chapter is overshadowed by the quantity and quality of the remaining content. The two geometric chapters, *Figures to be Divided* and *Pentominoes*, rely heavily upon cutting and tiling problems, respectively.

The final chapter, *Miscellaneous*, draws upon game contexts as sources of challenges. "Hopscotch on Prime Numbers" is one such game that struck me as being creative and playful. The challenge is to create a patio using consecutively numbered tiles such that it is possible to hop from the first row to the final row by a sequence of prime numbered tiles. Each hop must proceed to the next row and land upon a neighbouring tile. Consider the figure below with tiles numbered 1 to 30 on a patio having a width of 5 units. It would be possible to hop from 2 to 7 to 13 to 19 to 23 to 29, thus creating a path from the first to last row. Note that diagonal hops are permitted.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Suppose that there were 32 tiles rather than 30. This would require placing 31 and 32 in the final row (or altering the width of the patio). The final row may have less tiles. Completion of a path would become impossible because it would be necessary to get to 31, a tile accessible only from 26 or 27—neither of which is prime. Kurchan challenges solvers to find the minimal width (number of columns) that will allow one to hop from the first row to the final row using tiles numbered from 1 to 100.

Kurchan has created a delightful set of challenges that is well organized. The inclusion of solutions and a glossary (with select definitions and tables of perfect squares and primes) add to the value of the resource. Overall, I highly recommend this affordable book as a source of mathematical challenge and enjoyment for interested students, teachers, and armchair puzzlers.

On Some Examples of Geometric Fallacies

Toshio Seimiya

Euclid wrote a book containing a collection of geometric fallacies called Pseudaria. But this book was lost. Since Euclid's time many amusing examples of geometric fallacies have been published. Two well-known examples are the following ones that appeared in Rouse Ball's *Mathematical Recreations and Essays*.

(1) To prove that a right angle is equal to an angle that is greater than a right angle.

(2) To prove that every triangle is isosceles.

In these examples, when the figures are accurately drawn, the mistakes quickly become apparent.

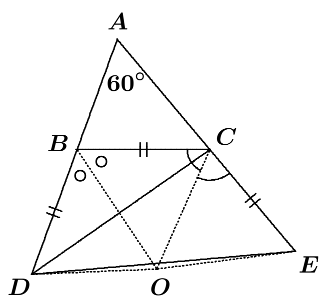
Even when the figures are accurately drawn and the argument is correct, geometric fallacies may occur. We often deduce false conclusions for lack of careful consideration. If the false conclusion is not absurd, it can easily be overlooked.

We propose some new examples of geometric fallacies. Errors in the fallacious geometric proofs will be explained afterwards in the Answer Section.

First we consider a (true) theorem, and then we investigate its converse. As is well known, the converse of a theorem is not always true. Geometric fallacies happen quite frequently in the attempted proof of the converse.

Theorem 1 Let ABC be a triangle with $\angle A = 60^\circ$. Let D be the point on AB produced beyond B such that $BD = BC$, and let E be the point on AC produced beyond C such that $BC = CE$. Then $\angle DBC = 2\angle CED$, $\angle BCE = 2\angle BDE$, and $\angle CDE = 30^\circ$.

Proof. Let O be the intersection of the bisectors of $\angle DBC$ and $\angle BCE$. Since $BD = BC = CE$, we have $\triangle BDO \equiv \triangle BCO$ and $\triangle BCO \equiv \triangle ECO$. Thus, $\angle BDO = \angle BCO$, $\angle BOD = \angle BOC$, $DO = CO$, and also $\angle CBO = \angle CEO$, $\angle BOC = \angle COE$. Since $\angle DBO = \angle CBO$ and $\angle BCO = \angle ECO$, we have



$$\begin{aligned}\angle BOC &= 90^\circ - \frac{1}{2}\angle BAC \\ &= 90^\circ - 30^\circ = 60^\circ .\end{aligned}$$

Hence, $\angle DOE = 3\angle BOC = 180^\circ$. Therefore, D, O, E are collinear. Thus,

$$\angle DBC \;=\; 2\angle CBO \;=\; 2\angle CEO \;=\; 2\angle CED$$

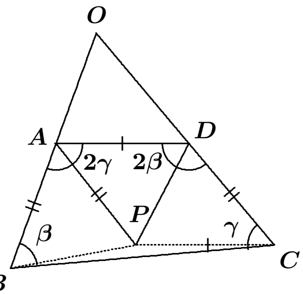
and

$$\angle BCE \;=\; 2\angle BCO \;=\; 2\angle BDO \;=\; 2\angle BDE\,.$$

Because $DO = CO$, we see that $\angle ODC = \angle OCD$. Therefore, $\angle CDE = \angle CDO = \frac{1}{2}\angle COE = 30^\circ$.

Next we shall consider two converses of this theorem.

Example 1. Let $ABCD$ be a convex quadrilateral with $AB = CD$. If we have $\angle A = 2\angle C$ and $\angle D = 2\angle B$, then $AB = AD$.



Proof. Let $\beta = \angle B$ and $\gamma = \angle C$. Then $\angle A = 2\gamma$ and $\angle D = 2\beta$. Since $\angle A + \angle B + \angle C + \angle D = 360^\circ$, we have $2\gamma + \beta + \gamma + 2\beta = 360^\circ$. Therefore, $\beta + \gamma = 120^\circ$. Let O be the point of intersection of AB and CD. Then

$$\angle BOC = 180^\circ - (\beta + \gamma) = 60^\circ\,.$$

Let P be the point such that $AP \parallel DC$ and $CP \parallel DA$. Since $APCD$ is a parallelogram, we have $AP = DC = AB$, and $PC = AD$, $\angle APC = \angle ADC = 2\beta$. Since $\angle BAP = \angle BOC = 60^\circ$ and $AB = AP$, we see that $\triangle ABP$ is equilateral. Thus, $\angle ABP = \angle APB = 60^\circ$. Since

$$\angle PBC \;=\; \angle ABC - \angle ABP \;=\; \beta - 60^\circ$$

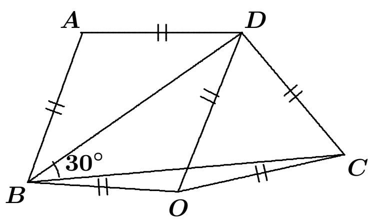
and

$$\angle BPC \;=\; 360^\circ - (\angle APB + \angle APC) \;=\; 360^\circ - (60^\circ + 2\beta)\,,$$

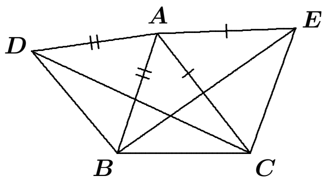
we have $\angle PCB = 180^\circ - (\angle PBC + \angle BPC) = \beta - 60^\circ$. Therefore, $\angle PBC = \angle PCB$, so that $PB = PC$. This implies $AB = AD$.

Example 2. Let $ABCD$ be a convex quadrilateral with $AB = AD = CD$. If $\angle DBC = 30^\circ$, then $\angle A = 2\angle C$.

Proof. Let O be the circumcentre of $\triangle BCD$. Then $OB = OC = OD$ and we have $\angle DOC = 2\angle DBC = 60^\circ$. Therefore, $\triangle OCD$ is equilateral, which means that $OB = OD = OC = CD$. Since $AB = AD = CD$, it follows that $AB = AD = OD = OB$; that is, $ABOD$ is a rhombus. Furthermore, $\angle BAD = \angle BOD = 2\angle BCD$.



Example 3. Suppose triangle ABC is given and equilateral triangles ABD and ACE are drawn outwardly on the sides AB and AC. If D, B, C, E are concyclic, then $AB = AC$.



Proof. Since D, B, C, E are concyclic, we see that $\angle BDC = \angle BEC$. Since $\angle BDA = \angle CEA = 60°$, it follows that

$$\begin{aligned} \angle ADC &= \angle BDA - \angle BDC \\ &= \angle CEA - \angle BEC \\ &= \angle AEB\,. \end{aligned} \tag{1}$$

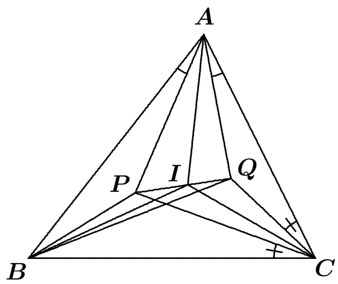
Since $AD = AB$, $AC = AE$, and $\angle DAC = \angle BAE$, it follows that $\triangle ADC \equiv \triangle ABE$, so that

$$\angle ADC = \angle ABE \tag{2}$$

From (1) and (2) we have $\angle AEB = \angle ABE$. Consequently, $AB = AE$. Since $AC = AE$, we have $AB = AC$.

Example 4. Suppose P and Q are two interior points of triangle ABC such that $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$. Suppose further that $AP : AQ = CP : CQ$. Then $BP : BQ = AP : AQ$.

Proof. Since $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$, we know that P and Q are isogonally conjugate points of $\triangle ABC$. Hence, $\angle ABP = \angle CBQ$. Let I be the intersection of PQ with the bisector of $\angle PAQ$. Then $PI : IQ = AP : AQ = CP : CQ$. Thus, $\angle PCI = \angle QCI$. Since $\angle PAB = \angle QAC$ and $\angle PCB = \angle QCA$, we have



$$\angle BAI = \angle PAB + \angle PAI = \angle QAC + \angle QAI = \angle CAI\,.$$

Similarly, we have $\angle BCI = \angle ACI$. Thus, I is the incentre of $\triangle ABC$, so that $\angle ABI = \angle CBI$. Therefore,

$$\angle PBI = \angle ABI - \angle ABP = \angle CBI - \angle CBQ = \angle QBI\,.$$

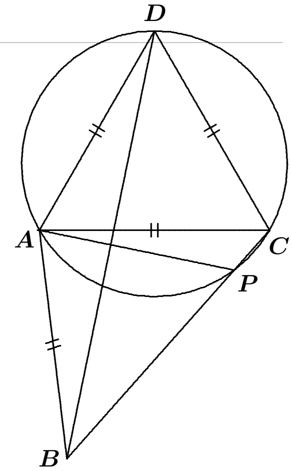
Hence, $BP : BQ = PI : IQ = AP : AQ$.

Remark. Notice in the above figure that $\dfrac{AP}{AQ} = \dfrac{CP}{CQ} > 1$, but $\dfrac{BP}{BQ} < 1$.

ANSWERS. The errors in the fallacious geometric proofs are briefly explained as follows.

Example 1. The mistake lies in the argument that if $\angle PBC = \angle PCB$, then $PB = PC$. If $\angle PBC = \angle PCB = 0°$, we cannot conclude that

$PB = PC$. Counterexample: suppose $ABCD$ is an isosceles trapezoid with $AB = CD \neq AD$ and $\angle A = \angle D = 120^\circ$.



Example 2. The mistake lies in the argument that we tacitly assumed that A and O are distinct points. There exists a case where A is the circumcenter of $\triangle BCD$. In this case, major angle BAD is equal to $2\angle BCD$. Counterexample: suppose $\triangle ACD$ is equilateral, and P is a point on the minor arc AC of the circumcircle of $\triangle ACD$ such that $AP > PC$. Let B be the point on CP produced beyond P such that $AB = AC$. Then $ABCD$ is a convex quadrilateral with $AB = AD = CD$, and $\angle DBC = 30^\circ$. In this case A is the circumcenter of $\triangle BCD$.

Example 3. The mistake lies in the argument that if $\angle AEB = \angle ABE$, then $AB = AE$. If $\angle AEB = \angle ABE = 0^\circ$, we cannot conclude that $AB = AE$. Counterexample: suppose ABC is a triangle with $\angle A = 120^\circ$ and $AB \neq AC$.

Example 4. It can easily be verified that $\angle ABP = \angle CBP$ and that $\angle ABQ = \angle CBQ$, so that B, P, Q, and I are collinear. The mistake lies in the argument that if $\angle PBI = \angle QBI$, then $BP : BQ = PI : IQ$. If $\angle PBI = \angle QBI = 0^\circ$, we cannot conclude that $BP : BQ = PI : IQ$.

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PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Jim Totten, Département de mathématiques et de statistique, University College of the Cariboo, Kamloops, BC V2C 5N3. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque ($\star$) *indique que le problème a été proposé sans solution.*

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}'' \times 11''$ *ou* A4, *la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er avril 2004**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format* « *epic* » *ou* « *eps* » (*encapsulated postscript*). *Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

2864. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Si a, b, c sont les côtés d'un triangle acutangle, montrer que

$$\sum_{\text{cyclique}} \sqrt{a^2+b^2-c^2}\sqrt{a^2-b^2+c^2} \;\le\; ab+bc+ca\,.$$

. . . .

If a, b, c are the sides of an acute angled triangle, prove that

$$\sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}\sqrt{a^2-b^2+c^2} \;\le\; ab+bc+ca\,.$$

2865. *Proposé par George Baloglou, SUNY Oswego, Oswego, NY, USA.*

Dans un triangle ABC donné, soit D, E et F les points d'intersections respectifs des droites *concourantes* AD, BE et CF avec les côtés d'un triangle donné ABC. Désignons respectivement par p_1 et p_2 les périmètres des triangles ABC et DEF, et par δ_1 et δ_2 leur aires. Montrer que

(a) $2p_2 \leq p_1$ si AD, BE et CF sont les bissectrices ;

(b) $2p_2 \leq p_1$ si AD, BE et CF sont les hauteurs ;

(c) $3p_2 \leq 2p_1$ pour tous les D, E, F si et seulement si le triangle ABC est équilatéral ;

(d) $4\delta_2 \leq \delta_1$ pour tous les D, E, F et quel que soit le triangle ABC.

. . . .

Suppose that D, E, F are the points at which the concurrent lines AD, BE, CF meet the sides of a given triangle ABC. Let p_1 and p_2 be the perimeters and δ_1 and δ_2 the areas of $\triangle ABC$ and $\triangle DEF$, respectively. Prove that

(a) $2p_2 \leq p_1$ if AD, BE, and CF are angle bisectors;

(b) $2p_2 \leq p_1$ if AD, BE, and CF are altitudes;

(c) $3p_2 \leq 2p_1$ for all D, E, F if and only if $\triangle ABC$ is equilateral;

(d) $4\delta_2 \leq \delta_1$ for all D, E, F and arbitrary $\triangle ABC$.

2866. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Pour deux cercles donnés Γ_1 et Γ_2, soit l et m les tangentes extérieures communes. Soit A et B, C et D les points de contacts respectifs de l et m avec Γ_1 et Γ_2. Désignons par M le point milieu du segment AB, et par P et Q les secondes intersections respectives de MC et MD avec Γ_1 et Γ_2.

Montrer que A, B, P and Q sont sur un même cercle.

. . . .

For two given circles Γ_1 and Γ_2, the lines l and m are external common tangents. The line l touches Γ_1 and Γ_2 at A and B, respectively, and the line m touches Γ_1 and Γ_2 at C and D, respectively. Suppose that M is the mid-point of the segment AB, and that P and Q are the second intersections of MC and MD with Γ_1 and Γ_2, respectively.

Prove that A, B, P, and Q are concyclic.

2867. *Proposé par Antreas P. Hatzipolakis et Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Etant donné deux points B et C, trouver le lieu du point A tel que le centre du cercle des 9-points du triangle ABC soit situé sur la droite BC.

. . . .

Given two points B and C, find the locus of the point A such that the centre of the nine-point circle of $\triangle ABC$ lies on the line BC.

2868. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans le triangle ABC, on a $c^4 = a^4 + b^4$.

(a) Montrer que ABC est un triangle acutangle.

(b) Trouver la fourchette de l'angle ACB.

(c)★ Comment peut-on généraliser au cas où $c^n = a^n + b^n$?

. . . .

In $\triangle ABC$, we have $c^4 = a^4 + b^4$.

(a) Show that $\triangle ABC$ is acute angled.

(b) Determine the range of $\angle ACB$.

(c)★ How can we generalize to $c^n = a^n + b^n$?

2869. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On donne un rectangle $ABCD$ d'aire S ; soit E et F situés respectivement sur les côtés AB et AD, de sorte que $[CEF] = \frac{1}{3}S$, où $[PQR]$ désigne l'aire du triangle PQR.

Montrer que l'angle ECF est plus petit ou égal à $\frac{\pi}{6}$.

. . . .

Given rectangle $ABCD$ with area S, let E and F be points on sides AB and AD, respectively, such that $[CEF] = \frac{1}{3}S$, where $[PQR]$ denotes the area of $\triangle PQR$.

Prove that $\angle ECF \leq \frac{\pi}{6}$.

2870. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un triangle ABC avec I comme centre du cercle inscrit, O comme centre de cercle circonscrit et G comme centre de gravité, on suppose que l'angle AIO est un angle droit. Montrer que IG est parallèle à BC.

. . . .

Given triangle ABC with incentre I, circumcentre O, and centroid G, suppose that $\angle AIO = 90^\circ$. Prove that $IG \| BC$.

2871. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Dans le triangle ABC, désignons les côtés par a, b et c, les symédianes par s_a, s_b et s_c, et le rayon du cercle circonscrit par R. Montrer que

$$\frac{bc}{s_a} + \frac{ca}{s_b} + \frac{ab}{s_c} \leq 6R.$$

. . . .

In $\triangle ABC$, denote the sides by a, b, c, the symmedians by s_a, s_b, s_c, and the circumradius by R. Prove that

$$\frac{bc}{s_a} + \frac{ca}{s_b} + \frac{ab}{s_c} \leq 6R.$$

2872. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On donne un triangle acutangle ABC d'orthocentre H et soit O le centre du cercle circonscrit ; on suppose que D est un point sur le côté AC avec $CD = 2AD$, que DO coupe BC en E, et que HO est parallèle à BC. Montrer que

(a) $DO = OE$, et (b) $DE = CE$.

. . . .

Given an acute angled triangle ABC with orthocentre H and circumcentre O, suppose that D is a point on the side AC such that $CD = 2AD$, that DO meets BC at E, and that $HO \parallel BC$. Prove that

(a) $DO = OE$, and (b) $DE = CE$.

2873. *Proposé par Kee-Wai Lau, Hong Kong, Chine.*

Trouver tous les nombres entiers positifs n tels que $x = y = z = 1$ soit l'unique solution du système d'équations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^n + y^n + z^n &= 3. \end{aligned}$$

. . . .

Find all positive integers n such that the system of equations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^n + y^n + z^n &= 3. \end{aligned}$$

has the unique solution $x = y = z = 1$.

2874. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Désignons par a, b et c les longueurs respectives des côtés BC, CA et AB d'un triangle ABC, et par s, r et R son demi-périmètre, le rayon du cercle inscrit et le rayon du cercle circonscrit. Soit $y = s/R$ et $x = r/R$.

Montrer que

1. $\sum\limits_{\text{cyclique}} \sin^2 A = 2 \iff y - x = 2 \iff \triangle ABC$ est rectangle ;
2. $\sum\limits_{\text{cyclique}} \sin^2 A > 2 \iff y - x > 2 \iff \triangle ABC$ est acutangle ;
3. $\sum\limits_{\text{cyclique}} \sin^2 A < 2 \iff y - x < 2 \iff \triangle ABC$ est obtusangle.

....

Let a, b, and c denote the side lengths BC, CA, and AB, respectively, of triangle ABC, and let s, r, and R denote the semi-perimeter, inradius, and circumradius of the triangle, respectively. Let $y = s/R$ and $x = r/R$.

Show that

1. $\sum\limits_{\text{cyclic}} \sin^2 A = 2 \iff y - x = 2 \iff \triangle ABC$ is right-angled;
2. $\sum\limits_{\text{cyclic}} \sin^2 A > 2 \iff y - x > 2 \iff \triangle ABC$ is acute-angled;
3. $\sum\limits_{\text{cyclic}} \sin^2 A < 2 \iff y - x < 2 \iff \triangle ABC$ is obtuse-angled.

2875. *Proposé par Michel Bataille, Rouen, France.*

On suppose que dans le triangle ABC le cercle inscrit est respectivement tangent aux côtés BC, CA et AB, en D, E et F.

Montrer que $EF^2 + FD^2 + DE^2 \leq \frac{s^2}{3}$, où s est le demi-périmètre du triangle ABC.

....

Suppose that the incircle of $\triangle ABC$ is tangent to the sides BC, CA, AB, at D, E, F, respectively.

Prove that $EF^2 + FD^2 + DE^2 \leq \frac{s^2}{3}$, where s is the semiperimeter of $\triangle ABC$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of LI ZHOU, Polk Community College, Winter Haven, FL, USA from the list of solvers of 2763 and 2764.

2508. [2000 : 46; 2001 : 58–61] *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan.*

(Corrected) In problem 2408 [1999 : 49; 2000 : 55] we defined a point P to be *Cevic* with respect to $\triangle ABC$ if the vertices D, E, F of its pedal triangle determine concurrent cevians; more precisely, D, E, F are the feet of perpendiculars from P to the respective sides BC, CA, AB, while AD, BE, CF are concurrent.

1. Show that a point D on the line BC can determine 0, 1, 2, or infinitely many positions for E on AC for which P is Cevic.
2. Describe the possible locations of E if D divides the segment BC in the ratio $\lambda : 1 - \lambda$ (when P is Cevic and λ is an arbitrary real number).

II. *A solution of O. Giering, Technische Universität München, München, Germany.*

By coincidence, while ***CRUX with MAYHEM*** readers were working on problem 2508, O. Giering was independently investigating what he calls *concurrence problems for triangles*: Kopunktalitätsprobleme bei Dreiecken, *Sitzungsber. Abt. II, Österreich Akad. Wiss., Math.-Naturwiss. Kl.* **209**, (2000), 3–18. Among the numerous results of that paper he shows that for a given triangle ABC, the locus of Cevic points is a cubic passing through the vertices, the orthocenter H, and the circumcentre O. Moreover, the cubic is centrally symmetric about O, while its asymptotes are the perpendicular bisectors of the sides. Giering also studies the mapping that associates with each Cevic point the intersection point of the three cevians; this mapping is easily seen to fix A, B, C, H, and to take O to the centroid G. Not so easily seen, the image of the cubic under this mapping is another cubic. Some of these results (and others not mentioned here) were discovered or proved with the help of a computer. Indeed, a major theme of the author is that the computer allows us today to extend the confines of "elementary geometry" beyond the study of lines and circles. Those ***CRUX with MAYHEM*** readers who do not have access to Giering's paper (and do not have a computer handy) can explore the special case of the problem where the triangle is isosceles: the cubic degenerates into a hyperbola together with the triangle's line of symmetry.

2766. [2002 : 397] *Proposed by K.R.S. Sastry, Bangalore, India.*

In a Heron triangle, the sides a, b, c satisfy the equation $b = a(a - c)$. Prove that the triangle is isosceles. (A Heron triangle has integer sides and integer area.)

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let a, b, c be the sides of *any* integer-sided triangle where $b = a(a-c)$. Let $a - c = d$. Then $b = ad$ and $c = a - d$. From the Triangle Inequality $a + c > b$. Hence,

$$\begin{aligned} ad &< 2a - d\,, \\ d &< \frac{2a}{a+1} < 2\,. \end{aligned}$$

Since d is an integer, $d = 1$, and $a = b$. Thus, the triangle is isosceles.

Such triangles exist with integer area. For example, $a = b = 17$ and $c = 16$.

Also solved by PIERRE BORNSZTEIN, Pontoise, France; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Both Bornsztein and the proposer noted that there are infinitely many such triangles.

2768. [2002 : 398] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let x_1, x_2, ..., x_n be n positive real numbers. Prove that

$$\frac{x_1}{\sqrt{x_1x_2 + x_2^2}} + \frac{x_2}{\sqrt{x_2x_3 + x_3^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1 + x_1^2}} \ \geq\ \frac{n}{\sqrt{2}}\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Define $f(t) = \dfrac{e^t}{\sqrt{e^t+1}}$ for $t \in \mathbb{R}$. Then $f'(t) = e^t\left(\frac{1}{2}e^t + 1\right)(e^t+1)^{-\frac{3}{2}}$ and $f''(t) = e^t\left(\frac{1}{4}e^{2t} + \frac{1}{2}e^t + 1\right)(e^t + 1)^{-\frac{5}{2}} > 0$. So f is strictly convex. Therefore,

$$\begin{aligned} &\frac{x_1}{\sqrt{x_1x_2 + x_2^2}} + \frac{x_2}{\sqrt{x_2x_3 + x_3^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1 + x_1^2}} \\ &\quad = f\left(\ln\frac{x_1}{x_2}\right) + f\left(\ln\frac{x_2}{x_3}\right) + \cdots + f\left(\ln\frac{x_n}{x_1}\right) \\ &\quad \geq nf\left(\frac{\ln\frac{x_1}{x_2} + \ln\frac{x_2}{x_3} + \cdots + \ln\frac{x_n}{x_1}}{n}\right) \\ &\quad = nf(0) = \frac{n}{\sqrt{2}}\,, \end{aligned}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL L. DAYAO, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There was one incomplete solution.

2769. [2002 : 398] Corrected in [2002 : 532] *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, suppose that $\cos B - \cos C = \cos A - \cos B \geq 0$. Prove that

$$\left(b^2+c^2\right)\cos A - \left(a^2+b^2\right)\cos C \;\leq\; \left(c^2-a^2\right)\sec B\,.$$

Solution by Michel Bataille, Rouen, France.

By hypothesis, $\cos A \geq \cos B \geq \cos C$, so that $A \leq B \leq C$. It follows that $B < \frac{\pi}{2}$; in particular, $\cos B > 0$.

By the Law of Sines, the proposed inequality is equivalent to

$$\begin{aligned} &2(\sin^2 B + \sin^2 C)\cos A\cos B \\ &\qquad -2(\sin^2 A + \sin^2 B)\cos C\cos B \\ &\qquad\qquad \leq\; 2(\sin^2 C - \sin^2 A)\,. \end{aligned} \tag{1}$$

Let $X = \cos B - \cos C = \cos A - \cos B$. Then

$$\begin{aligned} 2\cos A\cos B &= \cos^2 A + \cos^2 B - X^2\,, \\ 2\cos C\cos B &= \cos^2 C + \cos^2 B - X^2\,. \end{aligned}$$

Substituting in the left-hand side L of (1) yields

$$\begin{aligned} L &= (\sin^2 A - \sin^2 C)X^2 + \sin(B+A)\sin(B-A) \\ &\qquad + \sin(C+A)\sin(C-A) + \sin(C+B)\sin(C-B)\,, \\ &= (\sin^2 A - \sin^2 C)X^2 \\ &\qquad + \tfrac{1}{2}(\cos 2A - \cos 2B + \cos 2A - \cos 2C + \cos 2B - \cos 2C)\,, \\ &= (\sin^2 A - \sin^2 C)X^2 + 2(\sin^2 C - \sin^2 A)\,. \end{aligned}$$

Thus, (1) is equivalent to $\sin^2 A - \sin^2 C \leq 0$. Now by the observation above, $\sin^2 A - \sin^2 C = \cos^2 C - \cos^2 A = (\cos C - \cos A)(2\cos B) \leq 0$.

This completes the proof.

Also solved by EVANGELINE P. BAUTISTA and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2770. [2002 : 398] *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, MD, USA.*

In $\triangle ABC$, suppose that $a \leq b \leq c$ and $\angle ABC \neq \frac{\pi}{2}$. Prove that

$$2 + \sec B \;\leq\; \left(1 + \frac{b}{a}\right)\left(1 + \frac{b}{c}\right) .$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

We are given that $a \leq b \leq c$, which implies that $\angle ABC < \frac{\pi}{2}$. The proof is accomplished by using the following successive transformations of the desired inequality:

$$\begin{aligned}
\sec B &\leq \left(1 + \frac{b}{a}\right)\left(1 + \frac{b}{c}\right) - 2 \,, \\
\frac{1}{\cos B} &\leq \frac{ab + ac + b^2 + bc}{ac} - 2 \,, \\
\cos B &\geq \frac{ac}{ab + b^2 + bc - ac} \,, \\
\frac{a^2 + c^2 - b^2}{2ac} &\geq \frac{ac}{ab + b^2 + bc - ac} \,, \\
\frac{(a+b+c)(a-b+c)}{2ac} - 1 &\geq \frac{ac}{b(a+b+c) - ac} \,, \\
\frac{(a+b+c)(a-b+c)}{2ac} &\geq \frac{b(a+b+c)}{b(a+b+c) - ac} \,, \\
(a-b+c)[b(a+b+c) - ac] &\geq 2abc \,, \\
(a-b+c)b(a+b+c) - a^2c - abc - ac^2 &\geq 0 \,, \\
(a+b+c)(ab - b^2 + bc - ac) &\geq 0 \,, \\
(a+b+c)(b-a)(c-b) &\geq 0 \,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; WINFER TABARES and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2771★. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that

$$(a+b)^b \;=\; a^b + b^a \,.$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain (modified slightly by the editor).

Clearly, $(a, 1)$ is a solution for all positive integers a. We show that these are the only solution pairs.

Assuming that $b > 1$, we have

$$a^b + b^a \;=\; (a+b)^b \;=\; \sum_{k=0}^{b} \binom{b}{k} a^{b-k} b^k > a^b + b^b \,,$$

and thus, $a > b > 1$. Let $d = \gcd(a, b)$. Set $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$. Then we have $a_1 > b_1$ and $\gcd(a_1, b_1) = 1$, and the given equation becomes

$$\begin{aligned} d^b(a_1+b_1)^b &= d^b a_1^b + d^a b_1^a \,, \\ \text{or} \qquad (a_1+b_1)^b &= a_1^b + d^{a-b} b_1^a \,. \end{aligned} \tag{1}$$

If $d > 2$, then (1) has no solutions in positive integers by the Fermat-Wiles Theorem.

If $d = 2$, then (1) becomes

$$(a_1+b_1)^{2b_1} \;=\; a_1^{2b_1} + (2^{a_1-b_1} b_1^{a_1})^2 \,, \tag{2}$$

which implies that $a_1 + b_1$ and a_1 have the same parity. Thus, b_1 must be even. Let $b_1 = 2b_2$. Then (2) becomes

$$\left((a_1+b_1)^{b_2}\right)^4 \;=\; (a_1^{b_2})^4 + (2^{a_1-b_1} b_1^{a_1})^2 \,.$$

But it is well known that the equation $x^4 - y^4 = z^2$ has no non-trivial integer solutions. [*Ed:* See, for example, *Number Theory with Computer Applications* by Ramanujachary Kumanduri and Cristina Romero, p. 352.]

If $d = 1$, then $(a, b) = 1$ and the given equation can be written as

$$\begin{aligned} a^b + \binom{b}{1} a^{b-1} b + \cdots + \binom{b}{b-1} a b^{b-1} + b^b &= a^b + b^a \,, \\ \text{or} \quad a^{b-1}b^2 + \frac{b(b-1)}{2} a^{b-2} b^2 + \cdots + a b^b + b^b &= b^a \,. \end{aligned} \tag{3}$$

Suppose first that $b > 2$. Then $a > 3$, and (3) becomes

$$a^{b-1} + \frac{b(b-1)}{2} a^{b-2} + \cdots + a b^{b-2} + b^{b-2} \;=\; b^{a-2} \,. \tag{4}$$

If b has an odd prime divisor p, then $p \left| \frac{b(b-1)}{2} \right.$. Hence, (4) implies that $p | a^{b-1}$ and thus, $p | a$. However, this contradicts $(a, b) = 1$. Therefore, $b = 2^k$ where $k \in \mathbb{N}$. Since we are assuming that $b > 2$, we have $k > 1$

and $\frac{b(b-1)}{2} = 2^{k-1}(2^k - 1)$, which is even. Then (4) implies that a is even which again is a contradiction.

Hence, $b = 2$, and the given equation becomes $(a+2)^2 = a^2 + 2^a$, or $a + 1 = 2^{a-2}$. By simple induction on n, it is easily seen that $n \le 2^{n-3}$ for all integers $n \ge 6$, and that $n \ne 2^{n-3}$ for $1 \le n \le 5$. Hence, $a + 1 = 2^{a-2}$ has no solutions in integers. Our proof is now complete.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina. A partial solution was submitted by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Guersenzvaig considered a similar problem and showed that for positive integers a, b, and c, the Diophantine equation $(a+b)^b = a^b + b^{ac}$ holds if and only if either $b = 1$ or $(a, b, c) = (1, 2, 3)$. His proof also used the Fermat-Wiles Theorem.

2772. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f\big(yf(x) - 1\big) \;=\; x^2 f(y) - f(x) \text{ for all real } x \text{ and } y.$$

Solution by D. Kipp Johnson, Beaverton, OR, USA.

First we note that the constant function $f(x) = 0$ is a solution of the given equation.

Let $f(x)$ be a solution such that $f(x) \ne 0$ for some x. We show that this implies $f(x) = x$, thereby establishing that the given equation has only two solutions: $f(x) = 0$ and $f(x) = x$.

Letting $x = 0$ in the given equation gives

$$f(0)\,\big[f\big(yf(0) - 1\big) + 1\big] \;=\; 0\,.$$

Suppose $f(0) \ne 0$. Since $yf(0) - 1$ can take any value x (just replace y by $(x+1)/f(0)$), we obtain a possible candidate for a solution, namely, the constant function $f(x) = -1$. A quick check shows that this function is not a solution of the given equation. Therefore, $f(0) = 0$.

Now, suppose that $f(x) = 0$ for some $x \ne 0$. Then the original equation gives $0 = x^2 f(y)$, which can only happen if $f(y) = 0$ for all y, a contradiction, because we have already assumed that f is not zero everywhere. Thus, we can conclude that $f(x) = 0$ if and only if $x = 0$.

Letting $x = y = 1$ in the original equation gives $f(1)\, f\big(f(1) - 1\big) = 0$, and since $f(1) \ne 0$, we must have $f\big(f(1) - 1\big) = 0$, implying $f(1) - 1 = 0$. Hence, $f(1) = 1$.

When $x = 1$, the original equation then becomes

$$f(y - 1) \;=\; f(y) - 1\,. \tag{1}$$

Now, in the original equation, take $y = 1$ and use the equality (1) to obtain

$$\begin{aligned} f(x)\,[f(f(x)-1)] &= x^2 - f(x)\,, \\ f(x)\,[f(f(x))-1] &= x^2 - f(x)\,, \\ f(x)\,f(f(x)) - f(x) &= x^2 - f(x)\,. \end{aligned}$$

Thus,

$$f(x)\,f(f(x)) \;=\; x^2\,. \tag{2}$$

Now, replace x by $x-1$ in (2), apply (1) three times, and finally apply (2):

$$\begin{aligned} f(x-1)\,f(f(x-1)) &= (x-1)^2\,, \\ (f(x)-1)\,[f(f(x)-1)] &= (x-1)^2\,, \\ (f(x)-1)\,[f(f(x))-1] &= x^2-2x+1\,, \\ f(x)\,f(f(x)) - f(x) - f(f(x)) + 1 &= x^2-2x+1\,, \\ x^2 - f(x) - f(f(x)) + 1 &= x^2-2x+1\,. \end{aligned}$$

Therefore,

$$f(x) + f(f(x)) \;=\; 2x\,. \tag{3}$$

Now, (2) and (3) form a system of two equations in the unknowns $f(x)$ and $f(f(x))$. Eliminating $f(f(x))$ gives

$$[x - f(x)]^2 \;=\; 0\,,$$

so that $f(x) = x$, as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL DAYAO, Ateneo the Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution submitted.

2773. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China and Wu Kang, South China Normal University, Guang Zhou City, Guang Dong Province, China.*

Find all sequences of integers $x_1, x_2, \ldots, x_n, \ldots$, such that ij divides $x_i + x_j$ for any two distinct positive integers i and j.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

The only sequence is $x_n = 0$ for all n. By hypothesis, $1 \cdot n$ divides $x_1 + x_n$, and $2 \cdot n$ divides $x_2 + x_n$ for all positive integers $n > 2$. This implies that n divides the difference $x_1 + x_n - (x_2 + x_n) = x_1 - x_2$ for all such n. But the only integer divisible by an infinite set of integers is 0. Thus, $x_1 - x_2 = 0$. A similar argument shows that $x_k = x_{k+1}$ for all positive k, and all the terms of the sequence are equal.

Now $1 \cdot n$ divides $x_1 + x_n = 2x_1$ for all $n > 1$, which means that $x_1 = 0$, and all the terms are therefore zero.

Also solved by REY A. BARCELON and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL L. DAYAO, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2774. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Let x be a real number such that $0 < x \leq \frac{2}{9}\pi$. Prove that

$$(\sin x)^{\sin x} < \cos x .$$

(This is a generalization of Problem 10261 in the American Mathematical Monthly [1992 : 872, 1994 : 690]).

Solution by Kee-Wai Lau, Hong Kong, China.

For $0 < t < 1$, let $f(t) = 2t\ln t - \ln(1-t^2)$. Then by simple computations, $f''(t) = \dfrac{2}{t} + \dfrac{2(1+t^2)}{(1-t^2)^2} > 0$ and therefore, f is convex on $(0,1)$. Since $\lim\limits_{t\to 0^+} f(t) = 0$ by l'Hospital's Rule, and $f\left(\sin\dfrac{2\pi}{9}\right) \approx -0.035 < 0$, we see that $f(t) < 0$ for $0 < t < \sin\dfrac{2\pi}{9}$. Setting $t = \sin x$ we conclude that $2\sin x\ln(\sin x) - \ln(\cos^2 x) < 0$ or $(\sin x)^{\sin x} < \cos x$ for $0 < x < \dfrac{2\pi}{9}$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one partly incorrect solution.

Both Guersenvaig and Janous showed that the given inequality is, in fact, true for all x such that $0 < x < c$, where c is a constant with value slightly larger than $\frac{2\pi}{9} \approx 0.69813$. Guersenvaig gave $c \approx \frac{2.0392\pi}{9} \approx 0.711815108$ while Janous estimated (by "numeric methods") that $c \approx 0.71182794$.

2775. [2002 : 455] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

In $\triangle ABC$, let M be the mid-point of BC. Prove that

$$\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB) \geq 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) .$$

Solution by Michel Bataille, Rouen, France.

(a) Proof of $\cos\left(\frac{B-C}{2}\right) \;\geq\; \sin(\angle AMB)$.

Without loss of generality, suppose that $B \geq C$ (hence, $b \geq c$). Denote $AH = h$, $AD = d$, and $AM = m$ (the altitude, internal bisector and median from A to BC, respectively). Since $\angle BAH = \frac{\pi}{2} - B = \frac{A+C-B}{2} \leq \frac{A}{2}$ and $BD = \frac{ac}{b+c} \leq \frac{ac}{2c} = BM$, going along BC from B, we meet the points H, D, and M in that order. It follows that $\angle HAD = \frac{A}{2} - (\frac{\pi}{2} - B) = \frac{B-C}{2}$. Thus, $\cos\left(\frac{B-C}{2}\right) = \frac{h}{d}$, while $\sin(\angle AMB) = \frac{h}{m}$. Since $d \leq m$ (because $HD \leq HM$), we obtain $\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB)$.

There is equality if and only if $d = m$; that is, when $\triangle ABC$ is isosceles with $B = C$.

(b) Proof of $\sin(\angle AMB) \;\geq\; 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$.

The area of $\triangle AMB$ is $\frac{1}{2}\cdot\frac{a}{2}\cdot m\sin(\angle AMB)$ as well as $\frac{1}{2}\cdot\frac{a}{2}\cdot c\sin B$. Hence, $\sin(\angle AMB) = \frac{c\sin B}{m} = \frac{2R\sin B\sin C}{m}$. Thus, the proposed inequality is successively equivalent to

$$\begin{aligned}
R\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &\geq m\sin\left(\frac{A}{2}\right) \\
2R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &\geq m\sin A \\
\frac{R}{2}(\sin A + \sin B + \sin C) &\geq m\sin A \\
Rs &\geq am\,,
\end{aligned}$$

where R and s denote the circumradius and the semiperimeter of $\triangle ABC$, respectively. This completes the proof since the last inequality is known: two proofs (by Li Zhou and Mihàly Bencze) can be found in *Mathematics Magazine* Vol. 75 No 2, April 2002, pp. 148–9 as solutions to problem 1620.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2776. [2002 : 456] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, we have

(a) $a < b < c$,

(b) D is a point on the segment AC such that $CD = a$,

(c) E is a point on the segment AB such that $BE = a$,

(d) F is a point on CB produced such that $BF = b - a$,
(e) G is a point on the segment AB such that $AG = b$,
(f) H is a point on AC produced such that $AH = c$,
(g) K is a point on BC produced such that $BK = c$,
(h) M_1, M_2, and M_3, are the circumcentres of $\triangle ADE$, $\triangle BFG$ and $\triangle CHK$, respectively.

We denote the circumcentre by O, the circumradius by R, the incentre by I and the inradius by r. Prove that

(1) $DE \parallel FG \parallel HK$.
(2) $\triangle ADE$, $\triangle BFG$ and $\triangle CHK$ have equal circumradii, ρ.
(3) Show that $\rho = OI$.
(4) $DE : FG : HK = a : b : c$.
(5) Show that the circle centre I, radius R, passes through M_1, M_2, and M_3.

Editor's comment. Several readers added two further properties to our list:

(6) DE, FG, $HK \perp OI$.
(7) $M_1I \perp BC$, $M_2I \perp CA$, $M_3I \perp AB$.

I. *Solution to the original five parts by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(1) and (4). Since $\dfrac{DC}{AC} = \dfrac{a}{b} = \dfrac{BC}{FC}$, we have $DB \| AF$. Hence, $\angle DBE = \angle FAG$. Also, $\triangle BDE \sim \triangle AFG$ since $\dfrac{BD}{AF} = \dfrac{DC}{AC} = \dfrac{a}{b} = \dfrac{BE}{AG}$. Consequently, $DE \| FG$ and $\dfrac{DE}{FG} = \dfrac{a}{b}$. Similarly, $FG \| HK$ and $\dfrac{FG}{HK} = \dfrac{b}{c}$.

(2). By the Law of Sines, the circumradius of $\triangle ADE$ is

$$\frac{DE}{2\sin A} = \frac{a}{b} \cdot \frac{FG}{2\sin A} = \frac{\sin A}{\sin B} \cdot \frac{FG}{2\sin A} = \frac{FG}{2\sin B},$$

which is the circumradius of $\triangle BFG$. Similarly, $\triangle CHK$ also has the same circumradius ϱ.

(3). Let L and M be the mid-points of AB and AC, respectively. Suppose that the incircle of $\triangle ABC$ is tangent to AB at Z and to AC at Y (so that $IZ \perp AB$ and $IY \perp AC$). Locate S and T so that IZ and IY are the perpendicular bisectors of OS and OT, respectively. Then $OS \| AB$ and $OS = 2LZ = 2AZ - 2AL = (b + c - a) - c = b - a = AD$. Similarly, $OT \| AC$ and $OT = c - a = AE$. Thus, $\triangle OST \cong \triangle ADE$, whence $OI = AM_1 = \varrho$. (This argument provides an alternative proof to (2) without the use of trigonometry, since $BM_2 = CM_3 = OI$ by analogous arguments.)

(5) Let D', E', and M_1', be the reflections of D, E, and M_1 in the line AI. Then $\triangle OST$ is a translation of $\triangle AD'E'$. Hence, (recalling that I is the circumcentre of $\triangle OST$ because it was defined in (3) to be on the perpendicular bisectors of the sides), $R = OA = IM_1' = IM_1$. Similarly, $IM_2 = IM_3 = R$.

II. *Solution to parts* (6) *and* (7) *by Toshio Seimiya, Kawasaki, Japan.*

(6). Put $a + b + c = 2s$ and let the incircle be tangent to BC, CA, AB at X, Y, Z, respectively. Since $CD = CB$ and $CY = CX$, we get $DY = CD - CY = CB - CX = BX = s - b$. Similarly, we have $EZ = CX = s - c$. Since $IY \perp AC$, $IZ \perp AB$, and $IY = IZ$, we have

$$\begin{aligned} ID^2 - IE^2 &= (IY^2 + DY^2) - (IZ^2 + EZ^2) \;=\; DY^2 - EZ^2 \\ &= (s-b)^2 - (s-c)^2 \;=\; (2s-b-c)(c-b) \\ &= a(c-b)\,. \end{aligned} \tag{1}$$

The powers of D and of E with respect to the circumcircle of triangle ABC are $AD \cdot DC = OA^2 - OD^2$ and $AE \cdot EC = OA^2 - OE^2$. It follows that

$$\begin{aligned} OD^2 - OE^2 &= AE \cdot EB - AD \cdot DC \\ &= (c-a)a - (b-a)a \;=\; a(c-b)\,. \end{aligned} \tag{2}$$

From equations (1) and (2), we have $ID^2 - IE^2 = OD^2 - OE^2$, so that $IO \perp DE$. In the same way, we get $IO \perp FG$ and $IO \perp HK$. Of course, this argument also provides an alternative treatment of part (1).

(7). Using the powers of B and C with respect to circle ADE, we obtain $BE \cdot BA = BM_1^2 - AM_1^2$ and $CD \cdot CA = CM_1^2 - AM_1^2$. We therefore have

$$\begin{aligned} BM_1^2 - CM_1^2 &= (BM_1^2 - AM_1^2) - (CM_1^2 - AM_1^2) \\ &= BE \cdot BA - CD \cdot CA \;=\; ac - ab = a(c-b)\,. \end{aligned} \tag{3}$$

Since $IX \perp BC$, we have

$$BI^2 - CI^2 \;=\; BX^2 - CX^2 \;=\; (s-b)^2 - (s-c)^2 \;=\; a(c-b)\,. \tag{4}$$

Thus, from equations (3) and (4) we have $BM_1^2 - CM_1^2 = BI^2 - CI^2$, so that $M_1I \perp BC$. Likewise, $M_2I \perp CA$ and $M_3I \perp AB$.

*Also solved by *MICHEL BATAILLE, Rouen, France; *MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (omitted part* (5)*); JOEL SCHLOSBERG, student, New York University, NY, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. An asterisk indicates that the solver included a proof of parts* (6) *or* (7)*.*

Woo wondered how a configuration with so many noteworthy properties had not been discovered long ago. He added the suggestion that readers with suitable computer graphics might want to draw the figure so that the vertices of ABC are free to move around the circumcircle while it and the incircle remain fixed. The circles with centres M_i move in orbit about the incircle while triangle ABC changes its size and shape.

Comment based on the solution of Benito, Ciaurri, and Fernández. It follows from part (5) that the common ratio in part (4) is $\sqrt{1-\dfrac{2r}{R}}$:

$$\frac{DE}{a} = \frac{FG}{b} = \frac{HK}{c} = \frac{OI}{R} = \frac{\sqrt{R^2-2Rr}}{R} = \sqrt{1-\frac{2r}{R}}\,.$$

2777. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

For $x \geq 0$, let $y(x)$ represent the only real root of the equation $y^3 + 26xy = 27$. Prove that the function $x \mapsto y(x)$ is strictly decreasing on $[0,1]$. Also, find the value of $\displaystyle\int_0^1 y^2(x)\,\ln y(x)\,dx$.

Solution by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

We shall use the notation $y = f(x)$. We use the inverse function to show that the integral has the value $(243\cdot\ln 3 - 128)/52 \approx 2.67$.

For $y \neq 0$, we solve the equation $y^3 + 26xy = 27$ for x to get

$$x = g(y) = \frac{1}{26}\left(\frac{27-y^3}{y}\right)\,.$$

Since $x = g(y) < 0$ for $y < 0$ and we are only interested in $x \in [0,1]$, we look at values of y with $0 < y$. Now $g'(y) = -\dfrac{27+2y^3}{26y^2} < 0$ for $0 < y$, so that g is strictly decreasing (hence one-to-one) and differentiable (with non-zero derivative) on this interval. By inspection the ordered pairs $(1,1)$ and $(3,0)$ are on the graph of $x = g(y)$. Hence, the only part of the curve we care about is $1 \leq y \leq 3$. Restricting $g(y)$ to the interval $[1,3]$ gives a one-to-one, onto function $g : [1,3] \to [0,1]$. Therefore, the inverse $y = f(x)$, $f : [0,1] \to [1,3]$, exists and is also strictly decreasing, onto, and differentiable, with derivative being

$$f'(x) = \frac{1}{g'(f(x))} = -\frac{26[f(x)]^2}{27+2[f(x)]^3}\,.$$

In the integral, make the (one-to-one) substitution $y = f(x)$, so that

$$dy = f'(x)\,dx = -\frac{26[f(x)]^2}{27+2[f(x)]^3}\,dx\,,$$

which implies

$$-\frac{1}{26}(27+2y^3)\,dy = [f(x)]^2\,dx\,.$$

This substitution transforms the integral into

$$I = \int_0^1 [f(x)]^2 \ln[f(x)]\,dx = \frac{1}{26}\int_1^3 (27+2y^3)\ln y\,dy\,.$$

Using tables or integration by parts, we find

$$\begin{aligned} I &= \frac{1}{26}\left\{\left(27y+\frac{y^4}{2}\right)\ln y - 27y - \frac{y^4}{8}\right\}\Bigg|_1^3 \\ &= \frac{1}{52}\{243\cdot\ln 3 - 128\}\,. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; ROBERT BILINSKI, Outremont, QC (first part only); PAUL BRACKEN, Concordia University, Montréal, QC; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, La Seu d'Urgell, Spain; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Woo observed that one requires neither calculus nor any elaborate argument to show that $y(x)$ is strictly decreasing on $[0,1]$. The value $x = \dfrac{27-y^3}{26y}$ clearly decreases as y increases through its positive values because the numerator decreases while the denominator increases.

Since the problem stated that y is the only real root of the given equation for x a fixed non-negative number, there was no need for us to verify it. Nevertheless, many solvers did verify it, perhaps because uniqueness is critical for the integration. Descartes' Rule of Signs does the job simply. Alternatively, many solvers simply restricted the domain and range to the first quadrant as in the featured solution. The situation is different when x is a sufficiently large negative number; in that case the given cubic equation will be satisfied by two negative values of y in addition to its one positive value.

2778. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z \neq 1$ is a complex number such that $z^n = 1$ $(n \geq 1)$. Prove that

$$\big|nz - (n+z)\big| \;\leq\; \frac{(n+1)(2n+1)}{6}\,|z-1|^2\,.$$

Preliminary comment. The original proposal from Bencze was to show that

$$|nz - (n+2)| \;\leq\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

The editor mistakenly turned the first 2 into a z. Happily, the modified problem is still of interest, although it has lost some of its intuitive meaning.

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

Differentiating the familiar identity

$$\sum_{k=0}^{n} x^k \;=\; \frac{x^{n+1}-1}{x-1}$$

with respect to x, we get

$$\sum_{k=1}^{n} kx^{k-1} \;=\; \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}\,.$$

Multiplying both sides by x and differentiating again, we arrive at

$$\sum_{k=1}^{n} k^2x^{k-1} \;=\; g(x)\,,$$

where

$$g(x) \;=\; \frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}\,.$$

Taking $x=z$ and using $|z|=1$ (which we were given), we obtain

$$|g(z)| \;\le\; \sum_{k=1}^{n} k^2|z|^{k-1} \;=\; \frac{n(n+1)(2n+1)}{6}\,. \tag{1}$$

On the other side, taking into account that $z^n=1$, $z\neq 1$, we get

$$g(z) \;=\; \frac{n(nz^2-2(n+1)z+n+2)}{(z-1)^3} \;=\; \frac{n(nz-(n+2))}{(z-1)^2}\,. \tag{2}$$

From (1) and (2) we therefore conclude that

$$|nz-(n+2)| \;\le\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

This was the inequality that the proposer had intended for us to verify. For the inequality as it appears in our problem, it remains to show that

$$|nz-(n+z)| \;<\; |nz-(n+2)|\,,$$

where $|z|=1$ and n is a positive integer. This is a routine calculation, comparing the square of both sides and using $z\overline{z}=|z|^2=1\ge \mathrm{Re}(z)$. Alternatively, a sketch of the circles $|nz-(n+z)|$ and $|nz-(n+2)|$ (for $z=e^{it}$, $0\le t<2\pi$) makes the inequality clear. Note that the inequality is strict.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous proved the stronger inequality:

$$|nz-(n+z)| \;\le\; \frac{n^2}{3}|z-1|^2\,.$$

He pointed out that even his inequality is far from best possible, and conjectured that the best factor of $|z-1|^2$ on the right side would be

$$\frac{\sqrt{4n(n-1)\sin^2\left(\frac{\pi}{n}\right)+1}}{4\sin^2\left(\frac{\pi}{n}\right)}\,.$$

2779. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\alpha \in \mathbb{R}^*$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function for which $f(x) \neq \alpha x \ \forall\, x \in \mathbb{R}$. Prove that there is a sequence $\{x_n\}$ for which $\lim\limits_{n\to\infty} f'(x_n) = \alpha$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Put $g(x) = f(x) - \alpha x$. Then g is differentiable and $g(x) \neq 0$ for all $x \in \mathbb{R}$. It is clear that we need to show there exists a sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} g'(x_n) = 0$.

If $g'(c) = 0$ for some c, then we are done by taking $x_n \equiv c$. Now suppose that $g'(x) \neq 0$ for all $x \in \mathbb{R}$. We have four cases: (1) $g > 0$ and $g' > 0$; (2) $g > 0$ and $g' < 0$; (3) $g < 0$ and $g' > 0$; (4) $g < 0$ and $g' < 0$. In all cases, either $y_n = g(n)$ or $y_n = g(-n)$ is a bounded monotone sequence; thus, $\{y_n\}$ converges. By the Mean Value Theorem, there exists x_n in $(n, n+1)$ or $(-n-1, -n)$ such that $g'(x_n) = y_{n+1} - y_n$ or $g'(x_n) = y_n - y_{n+1}$. Since $\lim\limits_{n\to\infty} |y_{n+1} - y_n| = 0$, we have $\lim\limits_{n\to\infty} g'(x_n) = 0$.

[*Ed.* Most solvers noted that the result holds for $\alpha = 0$. Also, implicit in this solution, and explicitly stated in most solutions, is the fact that the Intermediate Value Theorem holds for derivatives.]

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HÉCTOR P. PÉREZ and NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer. There were 2 incorrect solutions.