

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2542. [2000 : 237] Proposed by Hassan A. Shah Ali, Tehran, Iran.

Suppose that k is a natural number and $\alpha_i \geq 0$, $i = 1, \dots, n$, and $\alpha_{n+1} = \alpha_1$. Prove that

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \alpha_i^{k-j} \alpha_{i+1}^{j-1} \geq \frac{k}{n^{k-2}} \left(\sum_{1 \leq i \leq n} \alpha_i \right)^{k-1}.$$

Determine the necessary and sufficient conditions for equality.

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and Henry Liu, Trinity College, Cambridge, England.

By the Power-Mean Inequality, if $a_i \geq 0$ for $i = 1, \dots, m$, and $r \in \mathbb{N} \cup \{0\}$, then

$$\sum_{i=1}^m a_i^r \geq \frac{1}{m^{r-1}} \left(\sum_{i=1}^m a_i \right)^r. \quad (1)$$

We have equality if and only if $r = 0$, or $r = 1$, or $r \geq 2$ and $a_1 = \dots = a_m$.

Next, we show that the given inequality holds for $n = 2$; that is, if $b_1, b_2 \geq 0$, then

$$\sum_{j=1}^k b_1^{k-j} b_2^{j-1} + \sum_{j=1}^k b_2^{k-j} b_1^{j-1} = 2 \sum_{j=1}^k b_1^{k-j} b_2^{j-1} \geq \frac{k}{2^{k-2}} (b_1 + b_2)^{k-1}. \quad (2)$$

We will use induction on k . If $k = 1$, then both sides of the inequality (2) are equal to 2 (or if $k = 2$, then both sides are equal to $2(b_1 + b_2)$). Suppose the inequality (2) holds for some $k \geq 2$. Multiplying (2) by b_1 , then by b_2 , and adding the two inequalities, we obtain

$$2 \sum_{j=1}^k b_1^{k+1-j} b_2^{j-1} + 2 \sum_{j=1}^k b_1^{k-j} b_2^j \geq \frac{k}{2^{k-2}} (b_1 + b_2)^{k-1} b_1 + \frac{k}{2^{k-2}} (b_1 + b_2)^{k-1} b_2$$

or

$$b_1^k + \sum_{j=2}^k b_1^{k+1-j} b_2^{j-1} + \sum_{j=1}^{k-1} b_1^{k-j} b_2^j + b_2^k \geq \frac{k}{2^{k-1}} (b_1 + b_2)^k.$$

By (1), $b_1^k + b_2^k \geq \frac{1}{2^{k-1}}(b_1 + b_2)^k$. Thus,

$$b_1^k + \sum_{j=2}^k b_1^{k+1-j} b_2^{j-1} + \sum_{j=1}^k b_1^{k-j} b_2^j + b_2^k + b_1^k + b_2^k \geq \frac{k+1}{2^{k-1}}(b_1 + b_2)^k$$

or

$$2 \sum_{j=1}^{k+1} b_1^{k+1-j} b_2^{j-1} \geq \frac{k+1}{2^{k-1}}(b_1 + b_2)^k.$$

The induction is complete, so that the inequality (2) is true. Equality in (2) holds if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $b_1 = b_2$. Applying (1) and (2), we have

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \alpha_i^{k-j} \alpha_{i+1}^{j-1} = \frac{1}{2} \sum_{i=1}^n \left(2 \sum_{j=1}^k \alpha_i^{k-j} \alpha_{i+1}^{j-1} \right)$$

$$\geq \frac{1}{2} \sum_{i=1}^n \left(\frac{k}{2^{k-2}} (\alpha_i + \alpha_{i+1})^{k-1} \right) \quad (3)$$

$$\geq \frac{k}{2^{k-1}} \left(\frac{1}{n^{k-2}} \left(\sum_{i=1}^n (\alpha_i + \alpha_{i+1}) \right)^{k-1} \right) \quad (4)$$

$$= \frac{k}{2^{k-1}} \left(\frac{2^{k-1}}{n^{k-2}} \left(\sum_{i=1}^n \alpha_i \right)^{k-1} \right)$$

$$= \frac{k}{n^{k-2}} \left(\sum_{i=1}^n \alpha_i \right)^{k-1},$$

as required.

The cases of equality are determined from (3): it holds if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $\alpha_1 = \cdots = \alpha_n$. (In (4), we have equality if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $\alpha_1 + \alpha_2 = \cdots = \alpha_n + \alpha_1$, which is a weaker condition.)

Also solved by AUSTRIAN IMO-TEAM 2000; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2543. Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.

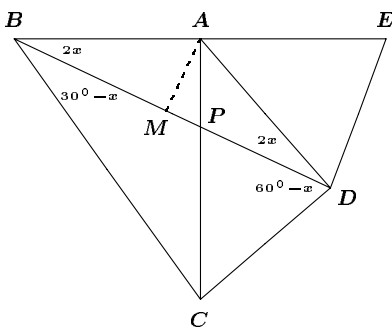
In quadrilateral $ABCD$, we have $\angle ABD = \angle ADB = \angle BDC = 40^\circ$, $\angle DBC = 10^\circ$ and AC and BD meet at P . Show that $BP = AP + PD$.

Editor's comment. The majority of solvers used the Sine Law (sometimes with the Cosine Law) to prove the result. A smaller number of solutions

relied only on elementary geometry to prove the result. We present the solution by T. Seimiya, who used elementary means to prove a more general result.

Solution by Toshio Seimiya, Kawasaki, Japan.

In quadrilateral $ABCD$, we have $\angle ABD = \angle ADB = 2x$, $\angle BDC = 60^\circ - x$, $\angle DBC = 30^\circ - x$ (where $0^\circ < x < 30^\circ$), and AC and BD meet at P . Show that $BP = AP + PD$. (When $x = 20^\circ$, we obtain problem 2543.)



Since $\angle ABD = \angle ADB$, we have that $AB = AD$. Let E be the point on BA produced beyond A such that $AE = AB = AD$. Then

$$\begin{aligned}\angle BED &= \frac{1}{2}\angle BAD \\ &= \frac{1}{2}(180^\circ - 2x - 2x) \\ &= 90^\circ - 2x.\end{aligned}$$

Since

$$\angle BCD = 180^\circ - (\angle CBD + \angle CDB) = 180^\circ - (90^\circ - 2x) = 90^\circ + 2x,$$

we have $\angle BED + \angle BCD = (90^\circ - 2x) + (90^\circ + 2x) = 180^\circ$.

Hence B, C, D and E lie on a circle.

Since $AB = AD = AE$, we see that A is the centre of this circle, and so $\angle CAD = 2\angle CBD = 2(30^\circ - x) = 60^\circ - 2x$.

$$\text{Thus, } \angle APB = \angle PAD + \angle ADB = (60^\circ - 2x) + 2x = 60^\circ.$$

Now let M be the foot of the perpendicular from A to BD . Since $AB = AD$, we see that M is the mid-point of BD .

$$\text{Hence, } \quad \quad \quad BP - PD = 2MP. \quad (1)$$

$$\text{Since } \angle APM = 60^\circ, \text{ we get } \quad AP = 2PM. \quad (2)$$

From (1) and (2), we deduce that $AP = BP - PD$, and therefore, that $BP = AP + PD$.

Also solved by HAYO AHLBURG, Benidorm, Spain; AUSTRIAN IMOTEAM 2000; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADIS, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; GERRY

LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2544. [2000 : 237] Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.

For any triangle ABC , find the exact value of

$$\sum_{\text{cyclic}} \frac{\cos A + \cos B}{1 + \cos A + \cos B - \cos C}.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

From

$$\begin{aligned} \frac{\cos A + \cos B}{\sin A + \sin B} &= \frac{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}} \\ &= \cot \left(\frac{A+B}{2} \right) = \tan \left(\frac{C}{2} \right) \end{aligned}$$

we get $\cos A + \cos B = (\sin A + \sin B) \cdot \tan \left(\frac{C}{2} \right)$.

Since $1 - \cos C = 2 \sin^2 \left(\frac{C}{2} \right) = \sin C \cdot \tan \left(\frac{C}{2} \right)$ and $\tan \left(\frac{C}{2} \right) \neq 0$, the given sum becomes

$$\sum_{\text{cyclic}} \frac{\sin A + \sin B}{\sin A + \sin B + \sin C} = 2.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RAJPARTAP KHANGURA, student, Angelo State University, San Angelo, TX, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VACLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; R. LAUMEN, Deurne-Antwerp, Belgium; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Visakhapatnam, India; HENRY PAN, Student, East York C.I. Toronto; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

About one third of the submitted solutions used the fact, with or without proof, that

$$\sum_{\text{cyclic}} \tan \left(\frac{A}{2} \right) \tan \left(\frac{B}{2} \right) = 1 \text{ when } A + B + C = \pi.$$

2545. [2000 : 237] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

For any triangle ABC , prove that

$$\sum_{\text{cyclic}} \frac{\sin^3 A}{\sin A (-\cos^2 A + \cos^2 B + \cos^2 C) + \sin B \cos(A - B) + \sin C \cos(A - C)} = 1.$$

Solution by Richard B. Eden, Ateneo de Manila University, Philippines.

We are going to use the identities $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$ and $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$.

First,

$$\begin{aligned} & \sin B \cos(A - B) + \sin C \cos(A - C) \\ &= \frac{1}{2} [\sin A + \sin(2B - A) + \sin A + \sin(2C - A)] \\ &= \sin A + [\sin(B + C - A) \cos(B - C)] \\ &= \sin A + \sin(180 - 2A) \cos(B - C) \\ &= \sin A + \sin 2A \cos(B - C) = \sin A [1 + 2 \cos A \cos(B - C)] \\ &= \sin A [1 - 2 \cos(B + C) \cos(B - C)] \\ &= \sin A [1 - \cos 2B - \cos 2C] \\ &= \sin A [3 - 2 \cos^2 B - 2 \cos^2 C]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{\text{cyclic}} \frac{\sin^3 A}{\sin A (-\cos^2 A + \cos^2 B + \cos^2 C) + \sin B \cos(A - B) + \sin C \cos(A - C)} \\ &= \sum_{\text{cyclic}} \frac{\sin^2 A}{(-\cos^2 A + \cos^2 B + \cos^2 C) + (3 - 2 \cos^2 B - 2 \cos^2 C)} \\ &= \sum_{\text{cyclic}} \frac{\sin^2 A}{3 - \cos^2 A - \cos^2 B - \cos^2 C} = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 A + \sin^2 B + \sin^2 C} = 1. \end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADIS, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RAJPARTAP KHANGURA, student, Angelo State University, San Angelo, TX, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ANDY LIU, University of Alberta, Edmonton, Alberta; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLU, Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most solutions were variations of the above one. Three solvers started with the Law of Sines. Four others gave abbreviated solutions, making use of the identity

$$\sum_{\text{cyclic}} \sin^2 A = 2(1 + \cos A \cos B \cos C).$$

2546. [2000 : 237] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Prove that triangle ABC is equilateral if and only if

$$a \cos(B - C) + b \cos(C - A) + c \cos(A - B) = \frac{a^4 + b^4 + c^4}{abc}.$$

Solution by Henry Liu, student, Trinity College, Cambridge, England.

Obviously, if $\triangle ABC$ is equilateral, then both sides of the given expression are equal to $3a$.

Conversely, suppose that the given equality holds. By repeated use of the AM-GM inequality,

$$\begin{aligned} a^4 + b^4 + c^4 &= \frac{a^4 + b^4}{2} + \frac{b^4 + c^4}{2} + \frac{c^4 + a^4}{2} \\ &\geq a^2 b^2 + b^2 c^2 + c^2 a^2 \\ &= a^2 \left(\frac{b^2 + c^2}{2} \right) + b^2 \left(\frac{c^2 + a^2}{2} \right) + c^2 \left(\frac{a^2 + b^2}{2} \right) \\ &\geq a^2 bc + b^2 ca + c^2 ab \\ &= abc(a + b + c). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{a^4 + b^4 + c^4}{abc} &\geq a + b + c \\ &\geq a \cos(B - C) + b \cos(C - A) + c \cos(A - B) \\ &= \frac{a^4 + b^4 + c^4}{abc}. \end{aligned}$$

Thus, we must have equality throughout. It is easy to see that

$$\frac{a^4 + b^4 + c^4}{abc} = a + b + c$$

if and only if $a = b = c$, and

$$a + b + c = a \cos(B - C) + b \cos(C - A) + c \cos(A - B)$$

if and only if $\cos(B - C) = \cos(C - A) = \cos(A - B) = 1$; that is, if and only if $A = B = C = \frac{\pi}{3}$. Therefore, $\triangle ABC$ is equilateral, which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ALEXANDER CORNELIUS, student, Angelo State University, San Angelo, TX, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Manila, the

Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulienengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; R. LAUMEN, Belgium; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most of the solvers showed that the given equality is equivalent to either

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 = 0$$

or some trigonometric equality, from which the result follows immediately. Lee seems to have found the fanciest equivalent form:

$$a(1 - \cos(B - C)) + b(1 - \cos(C - A)) + c(1 - \cos(A - B)) + \frac{1}{2abc} \sum_{\text{cyclic}} (a - b)^2[(a + b)^2 + c^2] = 0.$$

2547. [2000 : 238] Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.

In triangle ABC with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$, and area Δ , prove that

$$\frac{a^2 + b^2 + c^2}{\Delta} = 4\sqrt{7}.$$

Solution by Richard Eden, Ateneo de Manila University, Manila, the Philippines.

Let $\angle A = \frac{\pi}{7}$, $\angle B = \frac{2\pi}{7}$ and $\angle C = \frac{4\pi}{7}$. By the Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc \cos \frac{\pi}{7}.$$

Also, $\Delta = \frac{1}{2}bc \sin \frac{\pi}{7}$, so that

$$a^2 = b^2 + c^2 - 2 \left(\frac{2\Delta}{\sin \frac{\pi}{7}} \right) \cos \frac{\pi}{7} = b^2 + c^2 - 4\Delta \cot \frac{\pi}{7},$$

Similarly, $b^2 = c^2 + a^2 - 4\Delta \cot \frac{2\pi}{7}$

and $c^2 = a^2 + b^2 - 4\Delta \cot \frac{4\pi}{7}$.

Adding these three equalities and dividing by Δ yields,

$$\frac{a^2 + b^2 + c^2}{\Delta} = 4 \left(\cot \frac{\pi}{7} + \cot \frac{2\pi}{7} + \cot \frac{4\pi}{7} \right).$$

By problem 2537, $\cot \frac{\pi}{7} + \cot \frac{2\pi}{7} - \cot \frac{3\pi}{7} = \sqrt{7}$.

Since $\cot \frac{4\pi}{7} = -\cot(\pi - \frac{4\pi}{7})$, we obtain

$$\frac{a^2 + b^2 + c^2}{\Delta} = 4\sqrt{7}.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiá, Brazil; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian College, South Carolina, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA (two solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece (two solutions); D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2548*. [2000 : 238] Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana, USA.

Let $a(1) = 1$ and, for $n \geq 2$, define $a(n) = \lfloor a(n-1)/2 \rfloor$, if this is not in $\{0, a(1), \dots, a(n-1)\}$, and $a(n) = 3a(n-1)$ otherwise.

- (a) Does any positive integer occur more than once in this sequence?
 (b) Does every positive integer occur in this sequence?

Editorial comment.

There were no solutions received to either part. One reader (Richard Hess) reports that he has checked the first 15000 terms of the sequence by computer, and that there are no repeated terms and all positive integers up to 1435 appear. The problem remains open, and perhaps hopeless, though a solution to the first part may be possible.

The proposer also invites the readers to try replacing the divisor 2 and the multiplier 3 by any pair of relatively prime integers greater than 1, to see if the same behaviour seems to occur.

2549*. [2000 : 238] Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.

Is it possible to choose four points in the plane such that all the distances that they determine are odd integers?

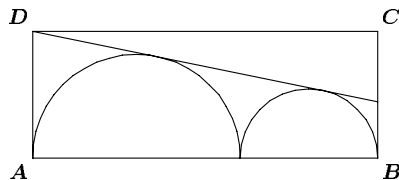
Editorial comment.

This problem appeared as problem B5 of the 1993 Putnam Competition. Thus a solution of it has been published in the October 1994 *American Mathematical Monthly*, pages 733–734. We thank John Leonard of the University of Arizona for this information. The proposer has noted that the problem follows from an even earlier *Monthly* article by Graham, Rothschild and Straus (“Are there $n + 2$ points in E^n with odd integral distances?”, Vol. 81(1974), pp. 21–25). Incidentally, according to this paper, the answer to the question in the title is “yes” if and only if $n \equiv -2 \pmod{16}$!

Solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JASON YOUNG, student, University of Arizona, Tucson, Arizona, USA. Young’s solution was found without knowledge of the problem’s earlier appearance in the Putnam.

2550. [2000 : 238] Proposed by Catherine Shevlin, Wallsend upon Tyne, England.

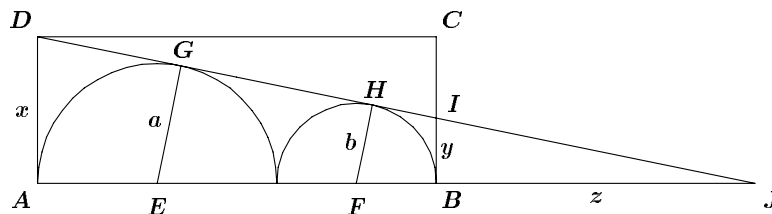
Given are two semi-circles, C_a and C_b of different radii a and b , and a rectangle $ABCD$ such that the diameters of the semi-circles lie contiguously on the side AB as shown, and the common tangent to the semi-circles passes through the vertex D of the rectangle.



Find, in terms of a and b , the ratio in which the common tangent divides the side BC .

Solution by Jesse Crawford, student, Angelo State University, San Angelo, TX, USA.

Let C_a and C_b be the semi-circles containing the points A and B , respectively, let E and F be the centres of C_a and C_b , respectively, and let G and H be the points of tangency of C_a and C_b with the common tangent line, respectively. Let I be the intersection of the common tangent with the line CB , and let J be the intersection of the extensions of DI and AB .



Since DJ is tangent to the semi-circles, we have $\angle EGJ = \angle FHJ = 90^\circ$. It follows that $\triangle ADJ \sim \triangle BIJ$ and $\triangle GEJ \sim \triangle HFJ$. Let x , y and z be the lengths of DA , IB and BJ , respectively. Using proportionality of the sides of similar triangles, we have

$$\frac{y}{x} = \frac{z}{2a + 2b + z}$$

and

$$\frac{a}{b} = \frac{a + 2b + z}{b + z}.$$

Eliminating z yields $\frac{y}{x} = \frac{b^2}{a^2}$. ■

Also solved by HAYO AHLBURG, Benidorm, Spain; CLAUDIO ARCONCHER, Jun-diaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, Quebec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADIS, Thessaloniki, Greece; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student, Cummer Valley Middle School, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEWAI LAU, Hong Kong; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; DANIEL REISZ, IREM, Université de Bourgogne, Vincelles, France; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; DARKO VELJAN, University of Zagreb, Zagreb, Croatia; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most solvers showed the equivalent result that the ratio of the two parts of CD was $\frac{a^2 - b^2}{b^2}$.

Diminnie noted that if $\frac{a}{b} = \phi$, the Golden Ratio $\frac{1+\sqrt{5}}{2}$, then $\frac{CI}{IB}$ is also the Golden Ratio.

2551. [2000 : 303] Proposed by Panos E. Tsaoussoglou, Athens, Greece.

Suppose that a_k ($1 \leq k \leq n$) are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

Prove that

$$\prod_{j=1}^n \sum_{k=1}^n e_{j,k} a_k^2 \geq \prod_{j=1}^n \left(\sum_{k=1}^n d_{j,k} a_k \right)^2.$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

On expanding out, the inequality reduces to

$$\prod_{k=1}^n [(n-2)S + a_k^2] \geq \prod_{k=1}^n (T - a_k)^2$$

where $S = \sum_{k=1}^n a_k^2$ and $T = \sum_{k=1}^n a_k$.

Since $(T - a_1)^2 \leq (n-1)(S - a_1^2)$, etc., it suffices to prove that

$$\prod_{k=1}^n [(n-2)S + a_k^2] \geq (n-1)^n \prod_{k=1}^n (S - a_k^2). \quad (1)$$

If we now let $x_k = S - a_k^2$, where $k = 1, 2, \dots, n$, so that

$$S = (x_1 + x_2 + \dots + x_n)/(n-1) \text{ and } a_k^2 = S - x_k,$$

(1) becomes

$$\prod_{k=1}^n (S' - x_k) \geq (n-1)^n x_1 x_2 \dots x_n, \text{ where } S' = x_1 + x_2 + \dots + x_n.$$

The result now follows by applying the AM-GM inequality to each of the factors $(S' - x_k)$ on the left-hand side. There is equality if and only if all the a_k 's are equal.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, Trinity College, Cambridge, England; and the proposer.

Janous notes that by employing the general Power-Mean Inequality, we can prove more: that for $\alpha \geq 1$

$$\prod_{j=1}^n \left(\sum_{k=1}^n d_{j,k} a_k \right)^\alpha \leq (n-1)^{n(\alpha-2)} \prod_{j=1}^n \left(\sum_{k=1}^n e_{j,k} a_k^\alpha \right).$$

2552. [2000 : 303] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Suppose that $a, b, c > 0$. If $x \geq \frac{a+b+c}{3\sqrt{3}} - 1$, prove that

$$\frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} \geq abc.$$

Solution by Michel Bataille, Rouen, France.

The condition on x can be rewritten as

$$(x + 1)^2 \geq \frac{(a + b + c)^2}{27}. \quad (1)$$

The AM–GM inequality gives

$$\frac{(a + b + c)^3}{27} \geq abc. \quad (2)$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left(\left(\frac{b + cx}{\sqrt{a}} \right)^2 + \left(\frac{c + ax}{\sqrt{b}} \right)^2 + \left(\frac{a + bx}{\sqrt{c}} \right)^2 \right) \left((\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2 \right) \\ & \geq ((b + cx) + (c + ax) + (a + bx))^2 \\ & = (a + b + c)^2 (x + 1)^2. \end{aligned}$$

Hence

$$\frac{(b + cx)^2}{a} + \frac{(c + ax)^2}{b} + \frac{(a + bx)^2}{c} \geq (a + b + c)(x + 1)^2.$$

Using (1) and (2), we obtain

$$\begin{aligned} \frac{(b + cx)^2}{a} + \frac{(c + ax)^2}{b} + \frac{(a + bx)^2}{c} & \geq (a + b + c) \frac{(a + b + c)^2}{27} \\ & \geq abc, \end{aligned}$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; HENRI LIU, Trinity College, Cambridge, England; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFLER, student, Gotham School, Bristol, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; IVAN SLAVOV, student, English Language High School, Stara Zagora, Bulgaria; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSOGLOU, Athens, Greece; and the proposer.

Most of the proposed solutions are similar to the one given above. Bencze, Seiffert and Slavov have suggested various generalizations of the given inequality.

2553. [2000 : 303] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find all real roots of the equation

$$\frac{\left(\sqrt{2x^2 - 2x + 12} - \sqrt{x^2 - 5} \right)^3}{(5x^2 - 2x - 3) \sqrt{2x^2 - 2x + 12}} = \frac{2}{9}.$$

I. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let x be a real root of the equation. Then

$$\frac{(u-v)^3}{(u^2+3v^2)u} = \frac{2}{9}, \quad (1)$$

where $u = \sqrt{2x^2 - 2x + 12}$ and $v = \sqrt{x^2 - 5}$. Obviously, $v \neq 0$ so that (1) may be written as

$$\frac{(w-1)^3}{(w^2+3)w} = \frac{2}{9},$$

where $w = u/v$. After clearing the fractions and expanding, this becomes

$$7w^3 - 27w^2 + 21w - 9 = 0,$$

or, equivalently,

$$(w-3)(7w^2 - 6w + 3) = 0.$$

The equation $7w^2 - 6w + 3 = 0$ has no real roots. Hence we must have $w = 3$. From $u = 3v$, we have $u^2 = 9v^2$ or $7x^2 + 2x - 57 = 0$. Thus, $x = -3$ and $x = 19/7$. It is easily verified that both of these are solutions, which means they are the only solutions.

II. *Solution by Michel Bataille, Rouen, France.*

Suppose that x is such a root and let $a = \sqrt{2x^2 - 2x + 12}$ and $b = \sqrt{x^2 - 5}$. Observing that $5x^2 - 2x - 3 = a^2 + 3b^2$, we get

$$\frac{2}{9} = \frac{(a-b)^3}{a(a^2+3b^2)} = \frac{2(a-b)^3}{(a+b)^3 + (a-b)^3}.$$

Hence $a \neq b$ and

$$1 + \left(\frac{a+b}{a-b}\right)^3 = 9,$$

from which $a = 3b$ is easily obtained. Since $a^2 = 9b^2$ may be written as $7x^2 + 2x - 57 = 0$, we must have $x = -3$ or $x = 19/7$. Conversely, when $x = -3$ or $x = 19/7$, it is easy to check that $2x^2 - 2x + 12$ and $x^2 - 5$ are positive or that $a = 3b$. This shows that these values are indeed solutions. In conclusion, there are exactly two real roots, namely -3 and $19/7$.

Also solved by ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD EDEN, Ateneo de Manila University, Manila, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; IVAN SLAVOV, Stara Zagora, Bulgaria; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLU, Athens, Greece; M^a JESÚS VILLAR RUBIO, Santander, Spain; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

2554. [2000 : 304] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In triangle ABC , prove that at least one of the quantities

$$\begin{aligned} & (a + b - c) \tan^2 \left(\frac{A}{2} \right) \tan \left(\frac{B}{2} \right), \\ & (-a + b + c) \tan^2 \left(\frac{B}{2} \right) \tan \left(\frac{C}{2} \right), \\ & (a - b + c) \tan^2 \left(\frac{C}{2} \right) \tan \left(\frac{A}{2} \right), \end{aligned}$$

is greater than or equal to $\frac{2r}{3}$, where r is the radius of the incircle of $\triangle ABC$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, (adapted by the editor).

We shall prove a stronger result; that is, the arithmetic mean, M , of the three given quantities is at least $2r/3$. Let $s = \frac{1}{2}(a + b + c)$ denote the semiperimeter of the triangle and let $x = s - a$, $y = s - b$, $z = s - c$. Then clearly x , y , and z are all positive with $x + y + z = s$.

By well-known formulas we have:

$$\begin{aligned} r &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{xyz}{x+y+z}}, \\ \tan \left(\frac{A}{2} \right) &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ etc.} \end{aligned}$$

Thus,

$$\begin{aligned} & (-a + b + c) \tan^2 \left(\frac{B}{2} \right) \tan \left(\frac{C}{2} \right) \\ &= 2(s-a) \frac{(s-c)(s-a)}{s(s-b)} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= \frac{2x^2z}{(x+y+z)y} \sqrt{\frac{xy}{(x+y+z)z}} = \frac{2x^2r}{y(x+y+z)}. \end{aligned}$$

Similarly,

$$(a - b + c) \tan^2 \left(\frac{C}{2} \right) \tan \left(\frac{A}{2} \right) = \frac{2y^2r}{z(x+y+z)}$$

and

$$(a + b - c) \tan^2 \left(\frac{A}{2} \right) \tan \left(\frac{B}{2} \right) = \frac{2z^2r}{x(x+y+z)}.$$

Hence, $M \geq \frac{2r}{3}$ is equivalent to

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z$$

which follows from the Cauchy-Schwarz Inequality since

$$\begin{aligned} \left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \right) (y + z + x) &\geq \left(\frac{x}{\sqrt{y}} \sqrt{y} + \frac{y}{\sqrt{z}} \sqrt{z} + \frac{z}{\sqrt{x}} \sqrt{x} \right)^2 \\ &= (x + y + z)^2. \end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, *Cala Figuera, Mallorca, Spain*; ŠEFKET ARSLANAGIĆ, *University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions)*; MICHEL BATAILLE, *Rouen, France*; RICHARD B. EDEN, *Ateneo de Manila University, Philippines*; MURRAY S. KLAMKIN, *University of Alberta, Edmonton, Alberta*; DAVID LOEFFER, *student, Cotham School, Bristol, UK*; JUAN-BOSCO ROMERO MÁRQUEZ, *Universidad de Valladolid, Valladolid, Spain*; HEINZ-JÜRGEN SEIFFERT, *Berlin, Germany*; ECKARD SPECHT, *Otto-von-Guericke University, Magdeburg, Germany*; IVAN SLAVOV, *student, Stara Zagora, Bulgaria*; PANOS E. TSAOUSSOGLOU, *Athens, Greece*; PETER Y. WOO, *Biola University, La Mirada, CA, USA*; and the proposer. There were one incorrect and one incomplete solutions.

Besides Janous, Woo also proved the stronger result by very similar argument.

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