An Interesting Arithmetic Problem

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In the textbook [1], one can find the following problem:

Let A, B, C, a, b, c, α, β, γ be distinct digits from 1, 2, 3, 4, 5, 6, 7, 8, 9. Give an example such that the equality below holds.

$$\begin{array}{r} A\,a\;\alpha \\ +\,B\,b\;\beta \\ \hline C\,c\;\gamma \end{array}$$

Students turned in many different results. Some of them gave even 20 solutions. The record in our class of thirty students is made by the second author of this article, who found 120 such examples.

Thus, the question is: How many such examples are there?

It is natural to check all possible choices of A, B, a, b, α, β from the nine digits 1, 2, $\dots$, 9, then add them together to find the solutions. That is to say, let A be one of the 9 digits, B be one of the remaining 8 digits, a be one of the remaining 7 digits, and so on. Then there are $9{\cdot}8{\cdot}7{\cdot}6{\cdot}5{\cdot}4 = 60480$ possible choices for checking. It is not reasonable to check by hand such a large amount of operations.

We will solve this problem by using some most basic counting techniques which could be useful for school teachers. The strategy to solve this complicated problem is to divide it into many small easier problems.

(1) Let us analyze one example to gain insight into the solution of the problem as instructed by George Pólya in [2].

$$\begin{array}{r} 2\;3\;5 \\ +\,7\;4\;6 \\ \hline 9\;8\;1 \end{array}$$

If we switch 2, 7, or 3, 4, or 5, 6 in the above example, we still have equality. Therefore, if one example is found, there are indeed $2^3 = 8$ examples. We will use the example with $A < B$, $a < b$, $\alpha < \beta$ as the representative of the group of eight examples.

(2) $A+B+C+a+b+c+\alpha+\beta+\gamma = 1+2+3+4+5+6+7+8+9 = 45$. Hence,

$$A + B + a + b + \alpha + \beta \;=\; 45 - (C + c + \gamma)\,.$$

(3) It is impossible that, in our problem, $A + B = C$, $a + b = c$ and $\alpha + \beta = \gamma$ hold simultaneously. For if so, we then have

$$\begin{aligned} A + B + a + b + \alpha + \beta &= C + c + \gamma\,, \\ 45 - (C + c + \gamma) &= C + c + \gamma\,, \\ C + c + \gamma &= \frac{45}{2}\,. \end{aligned}$$

This is a contradiction since the sum of three digits must be a positive integer.

(4) It is impossible that in our problem $A + B = C$, $a + b = 10 + c$ and $\alpha + \beta = 10 + \gamma$ hold simultaneously. For if so, we then have

$$\begin{aligned} A + B + a + b + \alpha + \beta &= 20 + (C + c + \gamma)\,, \\ 45 - (C + c + \gamma) &= 20 + (C + c + \gamma)\,, \\ C + c + \gamma &= \frac{25}{2}\,. \end{aligned}$$

This is a contradiction.

(5) Either (i) $A + B = C$, $a + b = c$ and $\alpha + \beta = 10 + \gamma$ or (ii) $A + B = C$, $a + b = 10 + c$ and $\alpha + \beta = \gamma$ holds. In each case, we prove that $C + c + \gamma = 18$.

Take, for instance, case (i): since $\alpha + \beta = 10 + \gamma$, when practicing the addition we have to add 1 unit to the second column, which we denote as follows

$$\begin{array}{r} A\,a\;\alpha \\ +\,B\,b.\beta \\ \hline C\,c\;\gamma \end{array}$$

Thus,

$$A + B = C\,, \qquad a + b + 1 = c\,, \qquad \alpha + \beta = 10 + \gamma\,.$$

Hence,

$$\begin{aligned} A + B + a + b + 1 + \alpha + \beta &= 10 + (C + c + \gamma)\,, \\ 45 - (C + c + \gamma) + 1 &= 10 + (C + c + \gamma)\,, \\ C + c + \gamma &= 18\,. \end{aligned}$$

Case (ii) can be similarly discussed.

(6) From the discussion (5), we know that, if an example is found, there is another example obtained by switching the order of the second and third columns. In the following expression, if the left is a solution, the right one is automatically a solution.

Since there is only one carry digit, as long as the column consisting of α, β, γ and the carry bit column stay together, the "summands" can be rearranged.

$$\begin{array}{r} A\,a\;\;\alpha \\ +\,B\,b.\,\beta \\ \hline C\,c\;\;\gamma \end{array} \qquad\qquad \begin{array}{r} a\;\alpha\;A \\ +\,b.\,\beta\;B \\ \hline c\;\gamma\;C \end{array}$$

For simplicity, we pick up the example where the third column satisfies $\alpha + \beta = 10 + \gamma$ as the representative. Combining with the discussion in (1), we may use one representative to represent a group of $8 \times 2 = 16$ examples.

(7) Now we discuss one possibility for $C + c + \gamma = 18$.

$1+8+9 = 18$. The remaining digits are $\{2, 3, 4, 5, 6, 7\}$. The possible choice of α, β from this remaining set of digits must be either (i) $4 + 7 = 11$ or (ii) $5 + 6 = 11$.

In case (i), since $2 + 6 = 8$, $3 + 5 = 8$ and $3 + 6 = 9$, we have three representative examples:

$$\begin{aligned} 234 + 657 &= 891\,, \\ 324 + 567 &= 891\,, \\ 324 + 657 &= 981\,. \end{aligned}$$

In case (ii), since $2 + 7 = 9$, we have one more representative example:

$$235 + 746 = 981\,.$$

(8) Other cases for $C + c + \gamma = 18$ are discussed as follows:

$2 + 7 + 9$ with remaining digits $\{1, 3, 4, 5, 6, 8\}$, possible choice of α, β is $4 + 8 = 12$, there are three representative examples:

$$\begin{aligned} 134 + 658 &= 792\,, \\ 214 + 758 &= 972\,, \\ 314 + 658 &= 972\,. \end{aligned}$$

$3 + 6 + 9$ with remaining digits $\{1, 2, 4, 5, 7, 8\}$, possible choice of α, β is $5 + 8 = 13$, there are two representative examples:

$$\begin{aligned} 215 + 478 &= 693\,, \\ 215 + 748 &= 963\,. \end{aligned}$$

$3 + 7 + 8$ with remaining digits $\{1, 2, 4, 5, 6, 9\}$, possible choice of α, β is $4 + 9 = 13$, there are three representative examples:

$$\begin{aligned} 124 + 659 &= 783\,, \\ 214 + 569 &= 783\,, \\ 214 + 659 &= 873\,. \end{aligned}$$

$4 + 5 + 9$ with remaining digits $\{1, 2, 3, 6, 7, 8\}$, since the last digit of the sum $\alpha + \beta$ must be either 4 or 5, the possible choices of α, β are (i) $6 + 8 = 14$, or (ii) $7 + 8 = 15$.

Case (i) gives two representative examples:

$$\begin{aligned} 216 + 378 &= 594\,, \\ 216 + 738 &= 954\,. \end{aligned}$$

Case (ii) gives two representative examples:

$$\begin{aligned} 127 + 368 &= 495\,, \\ 317 + 628 &= 945\,. \end{aligned}$$

$4 + 6 + 8$ with remaining digits $\{1, 2, 3, 5, 7, 9\}$, since the last digit of the sum $\alpha + \beta$ must be either 4 or 6, the possible choices of α, β are (i) $5 + 9 = 14$, or $7 + 9 = 16$.

Case (i) gives one representative example:

$$125 + 739 = 864\,.$$

Case (ii) gives two representative examples:

$$\begin{aligned} 127 + 359 &= 486\,, \\ 317 + 529 &= 846\,. \end{aligned}$$

$5 + 6 + 7$ with remaining digits $\{1, 2, 3, 4, 8, 9\}$, possible choice of α, β is $8 + 9 = 17$, there are two representative examples:

$$\begin{aligned} 128 + 439 &= 567\,, \\ 218 + 439 &= 657\,. \end{aligned}$$

(9) Conclusion: From the discussion of (7) and (8), there are 21 representatives. Therefore, there are altogether $16 \times 21 = 336$ examples.

The beauty of mathematics exists everywhere, even in simple arithmetic.

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References

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