THE OLYMPIAD CORNER

No. 215

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We start this number with a third set of Klamkin Quickies. Give them a try before looking forward to the solutions!

AND FIVE MORE KLAMKIN QUICKIES

1. A sphere of radius R is tangent to each of three concurrent mutually orthogonal lines. Determine the distance D between the point of concurrence and the centre of the sphere.

2. If $P(x, y, z, t)$ is a polynomial in x, y, z, t such that $P(x, y, z, t) = 0$ for all *real* x, y, z, t satisfying $x^2 + y^2 + z^2 - t^2 = 0$, prove that $P(x, y, z, t)$ is divisible by

$$x^2 + y^2 + z^2 - t^2 .$$

3. From a variable point P on a diameter AB of a given circle of radius r, two segments PQ and PR are drawn terminating on the circle such that the angles QPA and RPB are equal to a given angle θ. Determine the maximum length of the chord QR.

4. Using that $\dfrac{(rs)!}{s!(r!)^s}$ is an integer, where r, s are positive integers, prove that $\dfrac{(rst)!}{t!(s!)^t(r!)^{ts}}$ is an integer for positive integers r, s, t.

5. Determine the range of $\dfrac{\tan(x+y)}{\tan x}$ given that

$$\sin y \;=\; \sqrt{2}\sin(2x+y) .$$

Next we give the problems of the Swedish Mathematical Competition, Final Round 1997. My thanks go to Chris Small, Canadian Team Leader to the International Mathematical Olympiad in Romania for collecting them for use.

SWEDISH MATHEMATICAL COMPETITION 1997

Final Round

November 22, 1997 (Time: 5 hours)

1. Let AC be a diameter of a circle. Assume that AB is tangent to the circle at the point A and that the segment BC intersects the circle at D. Show that if $|AC| = 1$, $|AB| = a$ and $|CD| = b$ then

$$\frac{1}{a^2 + \frac{1}{2}} < \frac{b}{a} < \frac{1}{a^2}.$$

2. The bisector of the angle B in the triangle ABC intersects the side AC at the point D. Let E be a point on side AB such that $3\angle ACE = 2\angle BCE$. The segments BD and CE intersect at the point P. One knows that $|ED| = |DC| = |CP|$. Find the angles of the triangle.

3. Let the sum of the two integers A and B be odd. Show that any integer can be written in the form $x^2 - y^2 + Ax + By$, where x and y are integers.

4. A and B are playing a game consisting of two parts:

• A and B make one throw each with a die. If the outcome is x and y, respectively, a list is created consisting of all two-digit integers $10a + b$, with $a, b \in \{1, 2, 3, 4, 5, 6\}$ such that $10a + b \leq 10x + y$.

For instance, if $x = 2$ and $y = 3$ the list is:

$$11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 21,\ 22,\ 23.$$

• The players now reduce the number of integers in the list by replacing a pair of the integers in the list by the non-negative difference of the chosen integers. If A, for instance, chooses 14 and 21 in the above example these two integers are removed and replaced by the integer 7. The new list becomes:

$$7,\ 11,\ 12,\ 13,\ 15,\ 16,\ 22,\ 23.$$

In the next move B may choose, for instance, 7 and 23, reducing the number of integers by one, and leaving the list

$$11,\ 12,\ 13,\ 15,\ 16,\ 16,\ 22.$$

The game is over when the list has been reduced to only one integer.

If the integer in the final list has the same parity as the outcome of A's throw, then A is the winner. What is the probability that A wins the game?

5. Let $s(m)$ denote the sum of the digits of the integer m. Prove that for any integer n, with $n > 1$ and $n \neq 10$, there is a unique integer $f(n) \geq 2$ such that $s(k) + s(f(n) - k) = n$ for all integers k satisfying $0 < k < f(n)$.

6. Let M be a set of real numbers. Assume that M is the union of a finite number of disjoint intervals and that the total length of the intervals is greater than 1. Prove that M contains at least one pair of distinct numbers whose difference is an integer.

As a second set collected by Chris Small, Canadian Team Leader, we give selected problems of the Ukrainian Mathematical Olympiad 1998.

UKRAINIAN MATHEMATICAL OLYMPIAD 1998
Selected Problems
April, 1998

1. (9th grade) Prove the inequality

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \;\geq\; 3$$

for positive real numbers a, b, c, with $abc = 1$.

2. (9th grade) A convex polygon with 2000 vertices in a plane is given. Prove that we may mark 1998 points of the plane so that any triangle with vertices which are vertices of the polygon has exactly one marked point as an internal point.

3. (10th grade) Let M be an internal point on the side AC of a triangle ABC, and let O be the intersection point of perpendiculars from the mid-points of AM and MC to lines BC and AB respectively. Find the location of M such that the length of segment OM is minimal.

4. (11th grade) A triangle ABC is given. Altitude CD intersects the bisector BK of ABC, and the altitude KL of BKC, at the points M and N respectively. The circumscribed circle of BKN intersects segment AB at the point $P \neq B$. Prove that triangle KPM is isosceles.

5. (11th grade) For real numbers x, y, $z \in (0,1]$, prove the inequality

$$\frac{x}{1+y+zx}+\frac{y}{1+z+xy}+\frac{z}{1+x+yz} \;\leq\; \frac{3}{x+y+z}\,.$$

6. (11th grade) The function $f(x)$ is defined on $[0,1]$ and has values in $[0,1]$. It is known that $\lambda \in (0,1)$ exists such that $f(\lambda) \neq 0$ and $f(\lambda) \neq \lambda$. Also,

$$f(f(x)+y) \;=\; f(x)+f(y)$$

for all x and y from the range of definition of the equality.

(a) Give an example of such a function.

(b) Prove that for any $x \in [0,1]$,

$$\underbrace{f(f(\ldots f}_{19}(x)\ldots)) \;=\; \underbrace{f(f(\ldots f}_{98}(x)\ldots))\,.$$

7. (11th grade) Two spheres with distinct radii are externally tangent at point P. Line segments AB and CD are given such that the first sphere touches them at the points A and C, and the second sphere touches them at the points B and D. Let M and N be the orthogonal projections of the mid-points of segments AC and BD on the line joining the centres of the given spheres. Prove that $PM = PN$.

8 (11th grade) Let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of real numbers such that

$$x_1 = 1, \quad x_{n+1} = \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2, \quad n \geq 1.$$

Prove that

(a) $x_{n+1} \geq x_n$ for all $n \geq 4$;

(b) $[x_n] = n$ for all $n \geq 4$ ($[a]$ denotes the integer part of a).

Next we give the problems of the Vietnamese Mathematical Olympiad, Category A, 1998. Thanks again go to Chris Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 1998

Category A, Day 1

March 13, 1998 — Time: 3 hours

1. Let $a \geq 1$ be a real number. Define a sequence $\{x_n\}$ ($n = 1, 2, \ldots$) of real numbers by

$$x_1 = a, \; x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1 + \ln x_n}\right).$$

Prove that the sequence (x_n) has a finite limit, and determine it.

2. Let $ABCD$ be a tetrahedron and AA_1, BB_1, CC_1, DD_1 be diameters of the circumsphere of $ABCD$. Let A_0, B_0, C_0 and D_0 be the centroids of the triangles BCD, CDA, DAB and ABC, respectively. Prove that

(a) the lines A_0A_1, B_0B_1, C_0C_1 and D_0D_1 have a common point, which is denoted by F;

(b) the line passing through F and the mid-point of an edge is perpendicular to its opposite edge.

3. Let $\{a_n\}$ ($n = 0, 1, 2, \ldots$) be a sequence of positive integers defined recursively by

$$a_0 = 20, \; a_1 = 100, \; a_{n+2} = 4a_{n+1} + 5a_n + 20.$$

Determine the smallest positive integer h for which $a_{n+h} - a_h$ is divisible by 1998 for every non-negative integer n.

Category A, Day 2

March 14, 1998 — Time: 3 hours

4. Prove that there does not exist an infinite sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers satisfying the two following conditions simultaneously

$$|x_n| \leq 0.666 \quad \text{for } n = 1, 2, \ldots, \tag{1}$$

$$|x_n - x_m| \geq \frac{1}{n(n+1)} + \frac{1}{m(m+1)} \tag{2}$$

for all $m \neq n$ $(m, n = 1, 2, \ldots)$.

5. Determine the smallest possible value of the following expression

$$F(x, y) = \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$$

where x, y are real numbers.

6. Determine all positive integers n for which there exists a polynomial $P(x)$ with real coefficients satisfying

$$P(x^{1998} - x^{-1998}) = x^n - x^{-n}$$

for all real numbers $x \neq 0$.

As a final set to get your solving skills going, we give the problems of Category B of the Vietnamese Mathematical Olympiad 1998. Again, thanks go to Chris Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 1998

Category B, Day 1

March 13, 1998 — Time: 3 hours

1. Let a be a real number. Define a sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers by

$$x_1 = a, \quad x_{n+1} = \frac{x_n(x_n^2 + 3)}{3x_n^2 + 1}$$

for $n \geq 1$. Prove that the sequence has a finite limit, and determine it.

2. Let P be a point lying on a given sphere. Three mutually perpendicular rays from P intersect the sphere at points A, B and C. Prove that for all such triads of rays from P, the plane of the triangle ABC passes through a fixed point, and determine the largest possible value of the area of the triangle ABC.

3. Let a, b be integers. Define a sequence $\{a_n\}$ $(n = 0, 1, 2, \dots)$ of integers defined by

$$a_0 = a\,,\ a_1 = b\,,\ a_2 = 2b - a + 2\,,\ a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$$

for $n \geq 0$.

(a) Find the general term of the sequence.

(b) Determine all integers a, b, for which a_n is a perfect square for all $n \geq 1998$.

Category B, Day 2

March 14, 1998 — Time: 3 hours

4. Let $x_1, x_2, \dots, x_n$ $(n \geq 2)$ be real positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}\,.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998\,.$$

5. Determine the smallest possible value of the following expression

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-3)^2}$$

where x, y are real numbers such that $2x - y = 2$.

6. Prove that for each positive odd integer n there is exactly one polynomial $P(x)$ of degree n with real coefficients satisfying

$$P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n}$$

for all real $x \neq 0$.

Determine if the above assertion holds for positive even integers n.

Now we give Klamkin's answers for the five Quickies given at the start of this number.

SOLUTIONS TO "AND FIVE MORE KLAMKIN QUICKIES"

1. Let the three lines be the x, y, and z–axes of a rectilinear coordinate system and the equation of the sphere be $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$. The required distance squared is $a^2+b^2+c^2$. Since the distance from the centre of the sphere to each of the lines is R, we have

$$R^2 \;=\; b^2+c^2 \;=\; c^2+a^2 \;=\; a^2+b^2\,.$$

Hence, $D^2 = 3R^2/2$.

2. By the Remainder Theorem,

$$P(x,y,z,t) \;=\; (x^2+y^2+z^2-t^2)Q(x,y,z,t)+R(x,y,z)t+S(x,y,z)$$

where Q, R and S are polynomials. Now, letting t be successively $\pm(x^2+y^2+z^2)^{1/2}$, it follows that $R=S=0$.

3. Extend the chords QP and RP to intersect the circle again at points Q' and R'. It now follows that the arcs QR and $Q'R'$ are congruent and thus, their measures are $\pi-2\theta$. Then if O is the centre, triangle OQR is isosceles whose vertex angle is also $\pi-2\theta$. Hence, $QR = 2r\cos\theta$, which is the same for all P.

4. It follows from the given relation that both

$$\frac{(r(st))!}{(st)!(r!)^{st}} \quad\text{and}\quad \frac{(st)!}{t!(s!)^t}$$

are integers. Now just multiply them together.

5. Since

$$\begin{aligned}\frac{\sin y}{\sin(2x+y)} &= \frac{\sin(x+y)\cos x-\cos(x+y)\sin x}{\sin(x+y)\cos x+\cos(x+y)\sin x}\\ &= \frac{\tan(x+y)-\tan x}{\tan(x+y)+\tan x} \;=\; \sqrt{2}\,,\\ \text{so that}\quad \frac{\tan(x+y)}{\tan x} &= -3-2\sqrt{2} \;=\; \text{a constant.}\end{aligned}$$

Somehow we managed not to give the answers to the first five Klamkin Quickies given in the April Number of the *Corner*. Murray points out an error in the statement of problem 2.

SOLUTIONS TO FIVE KLAMKIN QUICKIES FROM APRIL 2001 CRUX with MAYHEM

1. Prove that

$$a+b+c \;\geq\; \sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\,,$$

where a, b, c are sides of a non-obtuse triangle.

Solution. By the power mean inequality

$$\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2} \;\leq\; 2c\,,$$

and similarly two other such inequalities. Then, adding, we get the desired result.

2. Determine the extreme values of the area of a triangle ABC, given the lengths of the two altitudes h_c, h_b. [Correction to question made.]

Solution. Let D and E be the feet of the altitudes h_b and h_c. Then by the Law of Sines applied to triangles ABD and ACE, $c=\frac{h_b}{\sin A}$ and $b=\frac{h_c}{\sin A}$. Twice the area is given by $2[ABC]=\frac{h_bh_c}{\sin A}$. Hence, the minimum area is $\frac{h_bh_c}{2}$ occuring for $A=\frac{\pi}{2}$. Also, by letting A approach π, the area becomes unbounded. In this case a would be arbitrarily large.

3. Determine the maximum area of a triangle ABC given the perimeter p and the angle A.

Solution. Since $2[ABC]=bc\sin A$, we have to maximize bc subject to

$$p \;=\; a+b+c, \quad \text{and} \quad a^2 \;=\; b^2+c^2-2bc\cos A\,.$$

Since $p-a=b+c\geq 2\sqrt{bc}$, bc will be a maximum when $b=c$ regardless of the value of a. Thus, we have

$$p \;=\; a+2b \quad \text{and} \quad a^2 \;=\; 2b^2-2b^2\cos A\,.$$

Then $(p-2b)^2=2b^2-2b^2\cos A$. Solving for b:

$$b \;=\; \frac{p\left(1+\sin\frac{A}{2}\right)}{2\cos^2\frac{A}{2}}$$

so that

$$\max[ABC] \;=\; \frac{p^2\left(\tan\frac{A}{2}\right)\left(1+\sin\frac{A}{2}\right)^2}{4\cos^2\frac{A}{2}}\,.$$

4. Determine the minimum value of

$$\sum\left[\frac{(a_2+a_3+a_4+a_5)}{a_1}\right]^{1/2}$$

where the sum is cyclic over the positive numbers a_1, a_2, a_3, a_4, a_5.

Solution. Applying the AM–GM Inequality to each term of the sum, the given sum is greater than or equal to

$$\sum 2\left(\frac{a_2a_3a_4a_5}{a_1^4}\right)^{1/8}$$

(where again the sum is cyclic). Finally applying the AM–GM Inequality again, the latter sum is greater than or equal to 10. There is equality in the given inequality if and only if the a_i's are equal. In a similar fashion it follows that if we increase the number of variables to $n+1$ and change the $1/2$ power to any positive number p, the minimum here would be $(n+1)^p$.

5. $ABCD$ and $AB'C'D'$ are any two parallelograms in a plane with A opposite to C and C'. Prove that BB', CC' and DD' are possible sides of a triangle.

Solution. Let the vectors from A to B and A to D be denoted by $\mathbf{U}_1$ and $\mathbf{U}_2$, and the vectors from A to B' and A to D' be denoted by $\mathbf{V}_1$ and $\mathbf{V}_2$. Then

$$\mathbf{BB'} = \mathbf{V}_1 - \mathbf{U}_1,\ \mathbf{DD'} = \mathbf{V}_2 - \mathbf{U}_2,\ \text{ and }\ \mathbf{CC'} = \mathbf{V}_1 + \mathbf{V}_2 - \mathbf{U}_1 - \mathbf{U}_2\,,$$

so that $\mathbf{CC'} = \mathbf{BB'} + \mathbf{DD'}$. The rest follows from the triangle inequality $|\mathbf{P}\pm\mathbf{Q}| \le |\mathbf{P}|+|\mathbf{Q}|$ and with equality only if $\mathbf{P}$ and $\mathbf{Q}$ have the same direction.

Remarks: Christopher J. Bradley of Clifton College, Bristol, UK, also submitted solutions for the quickies, noting a problem with problem 2. He points out that he gave problem 1 to a group of students in 1988.

Next a comment answering an editorial question we posed.

1. [2000 : 325] Let m and n be natural numbers such that m^2+n^2 divides into mn. Prove that $m=n$.

Comment by Achilleas Sinefakopoulos, student, University of Athens, Greece.

This should be read "mn divides m^2+n^2". The same problem was posed in TOURNAMEN 18, Spring 1997. (See the solution, page 157 in "International Mathematics Tournament of the Towns, 1993–1997", edited by P.J. Taylor and A.M. Storozhev, Australian Mathematics Trust, 1998, Australia.

It seems that a package of solutions from long-time contributor Miguel Amengual Covas, Cala Figuera, Mallorca, Spain must have gone astray. Last issue we corrected the record for some. In this number, I want to acknowledge that he sent solutions to two problems for which we gave solutions in the April number of the *Corner*: 4. [1999 : 199–200; 2001 : 181] and 7. [1999 : 199–200; 2001 : 183] of the Ninth Irish Mathematical Olympiad.

Next we continue with readers' solutions to the St. Petersburg City Mathematical Olympiad, Third Round [1999 : 262] which we began last issue.

ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

Third Round – February 25, 1996

11^{th} Grade (Time: 4 hours)

5. Find all quadruplets of polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ with real coefficients possessing the following remarkable property: for all integers x, y, z, t satisfying the condition $xy - zt = 1$, the equality $p_1(x)p_2(y) - p_3(z)p_4(t) = 1$ holds.

Solution by Pierre Bornsztein, Pontoise, France.

If p_1, p_2, p_3, p_4 are such polynomials, then for all $x \in \mathbb{Z}$:

$$p_1(x+1)p_2(1) - p_3(x)p_4(1) \;=\; 1 \qquad (1)$$

and

$$p_1(x+1)p_2(1) - p_3(1)p_4(x) \;=\; 1\,. \qquad (2)$$

Thus,

$$p_3(x)p_4(1) \;=\; p_3(1)p_4(x)\,. \qquad (3)$$

Moreover,

$$p_1(x)p_2(1) - p_3(x-1)p_4(1) \;=\; 1$$

and

$$p_1(1)p_2(x) - p_3(x-1)p_4(1) \;=\; 1\,.$$

Thus,

$$p_1(x)p_2(1) \;=\; p_1(1)p_2(x)\,. \qquad (4)$$

Moreover:

$$p_1(x)p_2(x) - p_3(x-1)p_4(x+1) \;=\; 1 \qquad (5)$$

and

$$p_1(x^2)p_2(1) - p_3(x-1)p_4(x+1) \;=\; 1\,.$$

Thus,

$$p_1(x^2)p_2(1) \;=\; p_1(x)p_2(x)\,. \qquad (6)$$

Since (3), (4), (5), (6) hold for an infinite number of values, and since they are relations between polynomials, then they hold for all real numbers x.

Case 1: If $p_1 \equiv 0$, then from (1):

$$p_3(x)p_4(1) = -1 \quad \text{for all } x \in \mathbb{R}\,.$$

It follows that p_3 is constant.

Since p_3 and p_4 are playing symmetric parts, we also have p_4 is constant. Then $p_3(x) = c$ and $p_4(x) = -\frac{1}{c}$ where $c \in \mathbb{R}^*$.

Conversely: $(0, p_2, c, -\frac{1}{c})$, where $p_2 \in \mathbb{R}[x]$ is arbitrary and $c \in \mathbb{R}^*$, is a solution of the problem.

By the same reasoning:

- $p_2 \equiv 0$ gives the solutions $(p_1, 0, c, -\frac{1}{c})$ where p_1 is an arbitrary polynomial, $p_1 \in \mathbb{R}[x]$, and $c \in \mathbb{R}^*$.
- $p_3 \equiv 0$ gives the solutions $(a, \frac{1}{a}, 0, p_4)$.
- $p_4 \equiv 0$ gives the solutions $(a, \frac{1}{a}, p_3, 0)$ where $a \in \mathbb{R}^*$ and p_3 (resp. p_4) is an arbitrary polynomial.

Now, we suppose that no one of p_1, p_2, p_3, p_4 is identically zero.

Case 2: If $p_1 \equiv a$ where $a \in \mathbb{R}^*$, then from (4), we have $p_2(x) = b$ where $b \in \mathbb{R}^*$.

If $p_3(1) = 0$, then, from (2) we have $ab = 1$. Thus, from (5), we have $p_3(x-1)p_4(x+1) = 0$ for all $x \in \mathbb{Z}$.

It follows that at least one of the polynomials p_3, p_4 has an infinite number of zeros. Thus, $p_3 \equiv 0$ or $p_4 \equiv 0$. Contradiction. We deduce that $p_3(1) \neq 0$.

Then, from (2), p_4 is constant: $p_4 \equiv d$, where $d \in \mathbb{R}^*$. In the same way, $p_3 \equiv c$, where $c \in \mathbb{R}^*$. And we must have $ab - cd = 1$.

Conversely: (a, b, c, d), where a, b, c, $d \in \mathbb{R}^*$ and $ab - cd = 1$, is a solution.

Notice that the condition a, b, c, d non-zero can be eased if $ab - cd = 1$ because it gives solutions of the first case.

Moreover, by the same reasoning, the cases p_3 is constant, p_4 is constant, p_2 is constant, give the same solutions.

Now, we suppose that:

Case 3: No one of p_1, p_2, p_3, p_4 is constant.

Then, for all $x \in \mathbb{Z}$: $p_1(x)p_2(0) - p_3(-1)p_4(1) = 1$ where p_1 is a non-constant polynomial. Thus, $p_2(0) = 0$.

In the same way, $p_1(0) = p_3(0) = p_4(0) = 0$. Then, we have $p_1(1)p_2(1) = p_1(1)p_2(1) - p_3(0)p_4(0) = 1$. It follows that $p_1(1) \neq 0$ and $p_2(1) \neq 0$.

In the same way, $-p_3(1)p_4(-1) = p_1(0)p_2(0) - p_3(1)p_4(-1) = 1$. Then $p_3(1) \neq 0$, $p_4(-1) \neq 0$.

But, it is clear that if (p_1, p_2, p_3, p_4) is a solution then $(ap_1, \frac{1}{a}p_2, bp_3, \frac{1}{b}p_4)$ is also a solution where a, b are arbitrary non-zero real numbers.

Then, with no loss of generality, we suppose that $p_1(1) = p_3(1) = 1$. Thus, $p_2(1) = 1$, $p_4(-1) = -1$.

From (4) and (6), we have $p_1(x^2) = p_1^2(x)$ for all $x \in \mathbb{R}$. We let $p_1(x) = \sum\limits_{i=0}^{n} a_i x^i$, $n \in \mathbb{N}^*$.

Identifying the coefficients, we have:

- $a_n = a_n^2$ with $a_n \neq 0$, then $a_n = 1$.
- $2a_n a_{n-1} = 0$, then $a_{n-1} = 0$.
- $2a_n a_{n-2} + a_{n-1}^2 = a_{n-1}$.

From the above, we have $a_{n-2} = 0$. And so on ... an easy induction leads to $a_k = 0$ for all $k < n$. Then $p_1(x) = x^n$ for some $n \in \mathbb{N}^*$.

From (4), we deduce that $p_2(x) = x^n$.

From (1), we have

$$(x+1)^n - 1 = p_3(x)p_4(1) = p_4(x)p_3(1) = p_4(x)\,.$$

Then,

$$p_4(x) = (x+1)^n - 1$$

and

$$p_3(x) = \frac{(x+1)^n - 1}{2^n - 1} \quad (\text{since } p_3(1) = 1)\,.$$

Then, for all x, y, z, t integers such that $xy - zt = 1$, we have

$$(xy)^n - \frac{((z+1)^n - 1)((t+1)^n - 1)}{2^n - 1} = 1\,.$$

Thus, for all $t \in \mathbb{Z}$, with $z = -1$, $x = 1$, $y = 1 - t$, we must have

$$(1-t)^n + \frac{(1+t)^n - 1}{2^n - 1} = 1\,.$$

It follows that the polynomial $Q(t) = (1-t)^n + \frac{(1+t)^n - 1}{2^n - 1}$ is constant. Since $n \geq 1$, the coefficient of t^n must be zero. That is,

$$(-1)^n + \frac{1}{2^n - 1} = 0\,.$$

Then,

$$2^n - 1 = 1 \quad \text{or} \quad 2^n - 1 = -1\,.$$

Thus, $n = 1$. It follows that

$$p_1(x) = p_2(x) = p_3(x) = p_4(x) = x\,.$$

Conversely: (x, x, x, x) is obviously a solution.

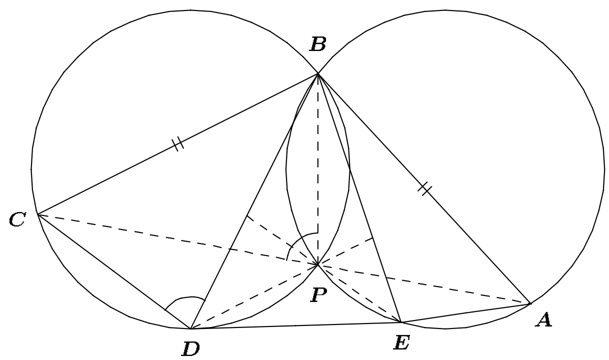
Conversely: The solutions are the quadruplets of the form:

(a) $\left(0, P, a, -\frac{1}{a}\right)$, $\left(P, 0, a, -\frac{1}{a}\right)$, $\left(a, \frac{1}{a}, P, 0\right)$, $\left(a, \frac{1}{a}, 0, P\right)$, where $P \in \mathbb{R}[x]$ and $a \in \mathbb{R}^*$ are arbitrary.

(b) (a, b, c, d), where $a, b, c, d \in \mathbb{R}$ with $ab - cd = 1$. This is, therefore, $\left(ax, \frac{1}{a}x, bx, \frac{1}{b}x\right)$ where $a, b \in \mathbb{R}^*$ are arbitrary.

6. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDE = 180^\circ$. Prove that the orthocentre of triangle BDE lies on diagonal AC.

Solution by Toshio Seimiya, Kawasaki, Japan.



(The condition $\angle AEB + \angle BDE = 180^\circ$ is incorrect. The correct condition is $\angle AEB + \angle BDC = 180^\circ$. After correcting, we shall solve the problem.)

Let P be the second intersection of the circumcircles of $\triangle BCD$ and $\triangle BAE$. Then

$$\angle APB = \angle AEB \quad \text{and} \quad \angle BPC = \angle BDC\,.$$

Therefore,

$$\angle APB + \angle BPC = \angle AEB + \angle BDC = 180^\circ\,.$$

Thus, A, P, C are collinear. That is, P is a point on the diagonal AC.

We put $\angle BAC = \angle BCA = \theta$ (because $AB = BC$). Then

$$\angle BEP = \angle BAP = \theta, \quad \text{and} \quad \angle BDP = \angle BCP = \theta\,.$$

Since $\angle ABE + \angle DBC = \angle EBD$, so that

$$\angle ABC = 2\angle EBD\,.$$

Since $\angle ABC + \angle BAC + \angle BCA = 180^\circ$, we have

$$2\angle EBD + 2\theta = 180^\circ\,.$$

Hence,

$$\angle EBD + \theta \;=\; 90^\circ\,.$$

Since $\angle EBD + \angle BDP = \angle EBD + \theta = 90^\circ$, and

$$\angle EBD + \angle BEP \;=\; \angle EBD + \theta \;=\; 90^\circ\,,$$

we have $DP \perp BE$ and $EP \perp BD$.

Hence, P is the orthocentre of $\triangle BDE$.

Thus, the orthocentre of $\triangle BDE$ lies on diagonal AC.

Now we turn to solutions to problems of the Selective Round, 11^{th} Grade of the St. Petersburg City Mathematical Olympiad [1999 : 263].

Selective Round – March 10, 1996

11^{th} Grade (Time: 5 hours)

1. It is known about real numbers $a_1, \ldots, a_{n+1}$; $b_1, \ldots, b_n$ that $0 \le b_k \le 1$ $(k = 1, \ldots, n)$ and $a_1 \ge a_2 \ge \cdots \ge a_{n+1} = 0$. Prove the inequality:

$$\sum_{i=1}^{n} a_k b_k \le \sum_{k=1}^{[\sum_{j=1}^{n} b_j]+1} a_k\,.$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's write-up.

Let $m = \left[\sum\limits_{j=1}^{n} b_j\right]$. If $m = 0$, then $0 \le \sum\limits_{j=1}^{n} b_j < 1$, so that

$$\sum_{k=1}^{n} a_k b_k \;\le\; a_1 \sum_{j=1}^{n} b_j \;\le\; a_1 \;=\; \sum_{k=1}^{m+1} a_k\,.$$

If $m = n$, then $b_k = 1$ $(k = 1, \ldots, n)$ and the inequality to be proved is obvious. Also, the case $n = 1$ is immediate, so that, in the following, we will suppose $n \ge 2$ and $1 \le m < n$.

Let $s(1)$ be the first integer ≥ 2 such that $\sum\limits_{j=1}^{s(1)} b_j \;\ge\; 1$,

$s(2)$ be the first integer $> s(1)$ such that $\sum\limits_{j=1}^{s(2)} b_j \;\ge\; 2$,

....

$s(m)$ be the first integer $> s(m-1)$ such that $\sum\limits_{j=1}^{s(m)} b_j \;\ge\; m$.

Note that $s(k) \geq k+1$, so that $a_{s(k)} \leq a_{k+1}$ $(k = 1, \ldots, m)$. Now,

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} c_k \text{ where } c_k = (a_k - a_{k+1}) \sum_{j=1}^{k} b_j \quad (\text{using } a_{n+1} = 0) \\ &= A_0 + A_1 + \cdots + A_m\end{aligned}$$

with

$$A_0 = \sum_{k=1}^{s(1)-1} c_k, \; A_1 = \sum_{k=s(1)}^{s(2)-1} c_k, \; \ldots, \; A_m = \sum_{k=s(m)}^{n} c_k \,.$$

Let $t \in \{0, 1, \ldots, m\}$. Then for $s(t) \leq k \leq s(t+1)-1$ (defining $s(0) = 1$, $s(m+1) = n+1$), we have $\sum_{j=1}^{k} b_j \leq t+1$ so that

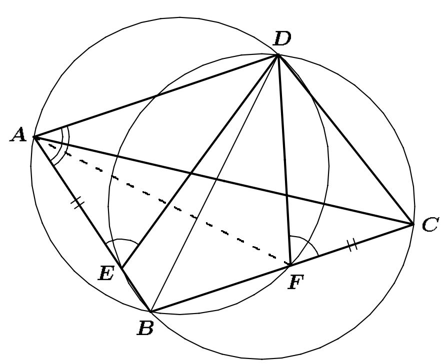
$$A_t \leq (t+1) \sum_{k=s(t)}^{s(t+1)-1} c_k = (t+1)(a_{s(t)} - a_{s(t+1)}) \,.$$

Thus, $A_0 \leq a_1 - a_{s(1)}$, $A_1 \leq 2(a_{s(1)} - a_{s(2)})$, $A_2 \leq 3(a_{s(2)} - a_{s(3)})$, $\ldots$, $A_m \leq (m+1)(a_{s(m)} - a_{n+1}) = (m+1)a_{s(m)}$, and, adding:

$$\sum_{k=1}^{n} a_k b_k \leq a_1 + a_{s(1)} + \cdots + a_{s(m)} \leq a_1 + a_2 + \cdots + a_{m+1} \,.$$

2. Segments AE and CF of equal length are taken on the sides AB and BC of a triangle ABC. The circle going through the points B, C, E and the circle going through the points A, B, F intersect at points B and D. Prove that the line BD is the bisector of angle ABC.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Since A, B, F, D are concyclic, we have

$$\angle EAD = \angle BAD = \angle CFD . \quad (1)$$

Similarly, we have

$$\angle AED = \angle FCD . \quad (2)$$

Since $AE = CF$, we get from (1) and (2) that $\triangle DAE \equiv \triangle DFC$. Therefore, $DA = DF$. Hence, we have

$$\angle ABD = \angle AFD = \angle FAD = \angle FBD = \angle CBD .$$

Thus, BD is the bisector of $\angle ABC$.

3. Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap+bq$ is also prime.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We use a solution derived from the similar solutions of Aassila and Evagelopoulos.

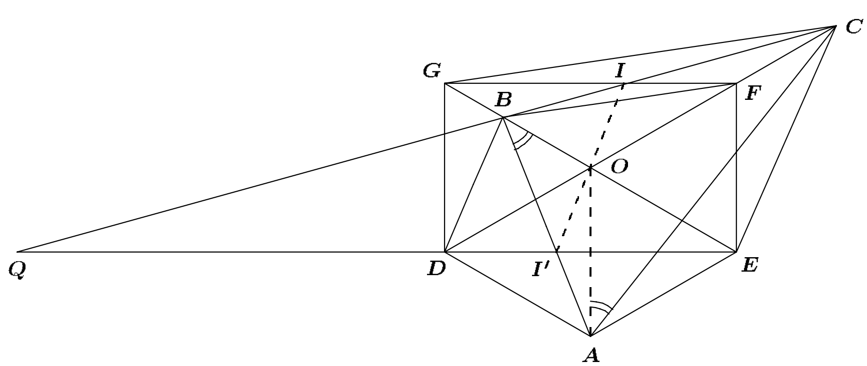
We are going to use the *reductio ad absurdum* method in order to prove the hypothesis of the problem.

Suppose there are positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

By Dirichlet's Theorem, there exist infinitely many primes in any non-zero residue class modulo m; this means there exists a pair p, q such that $p \equiv b \pmod{m}$, $q \equiv -a \pmod{m}$, giving $ap + bq$ is divisible by m, a contradiction.

5. In a triangle ABC the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. A point D is chosen on the half-line CO such that the triangle AOD is equilateral. The mid-perpendicular to the segment AO meets the line BC at point Q. Prove that the line OQ divides the segment BD into two equal parts.

Solutions by René Bornsztein, Antony, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of René Bornsztein.



We have $\angle BAC = 60°$, and

$$\angle BOC = \angle AOB = \angle AOC = 120° .$$

Then

$$\angle OBA = 60° - \angle BAO = \angle BAC - \angle BAO = \angle OAC .$$

Thus, $\triangle OAB$ and $\triangle OCA$ are similar. It follows that

$$\frac{OB}{OA} = \frac{OA}{OC} . \qquad (1)$$

Let E be the point such that $OEAD$ is a diamond. Let G, F be the points such that $EDGF$ is a rectangle, whose diagonals intersect with an angle of $120°$.

Then $G \in (OE)$, $F \in (OD)$, and $B \in (GE)$.

It follows that:

$$\frac{OB}{OE} = \frac{OB}{OA} = \frac{OA}{OC} = \frac{OD}{OC} \quad \text{(from (1).)}$$

Then (BD) is parallel to (EC).

Moreover,

$$\frac{OB}{OG} = \frac{OB}{OA} = \frac{OA}{OC} = \frac{OF}{OC} .$$

Then (BF) is parallel to (GC).

Let I, I' be respectively the intersection of (BC) and (GF), of (OI) and (ED). Since O is the centre of $DEFG$, O is the mid-point of (II').

From Thales' Theorem,

- In IGC and IFB: $\quad \dfrac{BF}{GC} = \dfrac{BI}{IC}$.
- In OBF and OGC: $\quad \dfrac{BF}{GC} = \dfrac{OB}{OG} = \dfrac{OB}{OE}$.

It follows that (OI) is parallel to (EC), and then to (BD).

Since Q, C, I, B are collinear, and Q, D, I' are collinear, since (OQ) divides (II') into two equal parts, then (OQ) divides (BD) into two equal parts.

Remark: (OQ) also divides (CE) into two equal parts.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations as well as Olympiad Contests!