THE ACADEMY CORNER

No. 42

Bruce Shawyer

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In this issue, we present problems of the Undergraduate Mathematics Competition held at Memorial University of Newfoundland on 27 March 2001. This competition was designed to be a little easier than usual, and some local high school students were invited to participate. Accordingly, we especially invite high school students everywhere to send us their solutions to these problems. A copy of *Inequalities* by E.J. Barbeau and B.L.R. Shawyer (in the **ATOM** series) will be awarded to the best solution set sent in by a high school student. The deadline to receive such solutions is 14 December 2001. High school students should enclose a note from a teacher verifying that they are indeed enrolled in a high school on the date on which the solutions are mailed. (No FAX or email entries accepted.)

Memorial's Local Undergraduate Competition Winter 2001

1. When $x^3 + px + 5$ is divided by $x - 1$, the remainder is the same as when it is divided by $x + 1$. Find p.

2. Two astronauts, Pat and Chris, are orbiting the earth (in circular orbits) in separate capsules. They are orbiting in the same direction along the equator. Pat orbits in 3 hours and Chris in $7\frac{1}{2}$ hours. At 12 noon Chris sees Pat directly below. How long will it be before they are one above the other again?

3. Given $\triangle ABC$ with right angle C and leg lengths a, b. From a point P on AB, perpendiculars are drawn to meet AC at S and BC at T. Find the minimum possible length of ST.

4. A unit square and a unit equilateral triangle share an edge. There is a unique circle that passes through the vertex of the triangle and two vertices of the square that are not on the shared edge. Determine the radius of the circle.

   [*This problem is due to E.J. Barbeau, University of Toronto, Toronto, Ontario.*]

5. Show that
$$\frac{\sqrt{y^2+1}+y+x}{x\sqrt{y^2+1}-xy+1}$$
   is independent of x, assuming that the denominator is not equal to zero.

6. Let P_n be the number of permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, \ldots, n$, with the following property: there exists exactly one index $i \in \{1, \ldots, n-1\}$ such that, $a_i > a_{i+1}$.

   (a) Find P_2, P_3 and P_4.

   (b) Show that $P_n = 2^n - n - 1$, $(n > 1)$.

   [*This is a modified version from the national Bulgarian Olympiad of 1995.*]

Next, we present some more solutions to problems of the 2000 Atlantic Provinces Council on the Sciences Mathematics Competition [2000 : 450, 2001 : 86]. These solutions were sent in by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

2000 Atlantic Provinces Council on the Sciences Mathematics Competition

5. The three-term geometric progression $(2, 10, 50)$ is such that

$$(2+10+50)*(2-10+50) \;=\; 2^2+10^2+50^2\,.$$

   (a) Generalize this (with proof) to other three-term geometric progressions.

   (b) Generalize (with proof) to geometric progressions of length n.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) In general, for three-term geometric progressions, we have

$$\begin{aligned}&\left(a+ar+ar^2\right)\left(a-ar+ar^2\right)\\ &\quad=\;\left(a+ar^2\right)^2-a^2r^2\\ &\quad=\;a^2+a^2r^2+a^2r^4 \;=\; a^2+(ar)^2+(ar^2)^2\,.\end{aligned}$$

(b) For **odd** n, consider a geometric progression of length n, $(a, ar, ar^2, \ldots, ar^{n-1})$. If $r \neq \pm 1$, we have

$$\begin{aligned}&\left(a+ar+ar^2+\cdots+ar^{n-1}\right)\left(a-ar+ar^2-\cdots+ar^{n-1}\right)\\ &\qquad=\;\frac{a\left(1-r^n\right)}{1-r}\times\frac{a\left(1+r^n\right)}{1+r}\\ &\qquad=\;a^2\left(1+r^2+r^2\cdots+r^{2n-2}\right)\\ &\qquad=\;a^2+(ar)^2+(ar^2)^2+\cdots+(ar^{n-1})^2\,. \qquad (1)\end{aligned}$$

It can also readily be verified that when $r = \pm 1$, both sides of (1) equal na^2, and, hence, (1) holds for all n.

We note that the conclusion does not hold for even n; for example, for $n = 2$, we have

$$(a + ar)(a - ar) \;=\; a^2 - a^2r^2 \;\neq\; a^2 + (ar)^2$$

in general.

7. Without calculator or elaborate computation, show that

$$3^{2701} \equiv 3 \pmod{2701}\,.$$

NOTE: $2701 = 37 \times 73$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Note first that $2701 = 37 \times 73$ and that both 37 and 73 are primes.

By Fermat–Euler's Theorem, we have $3^{37} \equiv 3 \pmod{37}$ and $3^{73} \equiv 3 \pmod{73}$. Hence,

$$3^{2701} = \left(3^{37}\right)^{73} \equiv 3^{73} \pmod{37}\,, \tag{1}$$

$$3^{2701} = \left(3^{73}\right)^{37} \equiv 3^{37} \pmod{73}\,. \tag{2}$$

By Fermat's Little Theorem, we have $3^{36} \equiv 1 \pmod{37}$, and further, $3^{72} \equiv 1 \pmod{37}$. Hence,

$$3^{73} \equiv 3 \pmod{37}\,. \tag{3}$$

Also, from $3^4 = 81 \equiv 8 \pmod{73}$, we have $3^6 \equiv 72 \equiv -1 \pmod{73}$, so that $3^{36} \equiv 1 \pmod{73}$. Hence,

$$3^{37} \equiv 3 \pmod{73}\,. \tag{4}$$

From (1) and (3), we have $3^{2701} \equiv 3 \pmod{37}$.

From (2) and (4), we have $3^{2701} \equiv 3 \pmod{73}$.

Finally, since $\gcd(37, 73) = 1$, we have $3^{2701} \equiv 3 \pmod{2701}$.

Wang also sent in a solution to problem 2 — we have already published a solution in [2001 : 86].

THE OLYMPIAD CORNER

No. 215

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We start this number with a third set of Klamkin Quickies. Give them a try before looking forward to the solutions!

AND FIVE MORE KLAMKIN QUICKIES

1. A sphere of radius R is tangent to each of three concurrent mutually orthogonal lines. Determine the distance D between the point of concurrence and the centre of the sphere.

2. If $P(x,y,z,t)$ is a polynomial in x, y, z, t such that $P(x,y,z,t)=0$ for all *real* x, y, z, t satisfying $x^2+y^2+z^2-t^2=0$, prove that $P(x,y,z,t)$ is divisible by

$$x^2+y^2+z^2-t^2\,.$$

3. From a variable point P on a diameter AB of a given circle of radius r, two segments PQ and PR are drawn terminating on the circle such that the angles QPA and RPB are equal to a given angle θ. Determine the maximum length of the chord QR.

4. Using that $\dfrac{(rs)!}{s!(r!)^s}$ is an integer, where r, s are positive integers, prove that $\dfrac{(rst)!}{t!(s!)^t(r!)^{ts}}$ is an integer for positive integers r, s, t.

5. Determine the range of $\dfrac{\tan(x+y)}{\tan x}$ given that

$$\sin y \;=\; \sqrt{2}\sin(2x+y)\,.$$

Next we give the problems of the Swedish Mathematical Competition, Final Round 1997. My thanks go to Chris Small, Canadian Team Leader to the International Mathematical Olympiad in Romania for collecting them for use.

SWEDISH MATHEMATICAL COMPETITION 1997

Final Round

November 22, 1997 (Time: 5 hours)

1. Let AC be a diameter of a circle. Assume that AB is tangent to the circle at the point A and that the segment BC intersects the circle at D. Show that if $|AC| = 1$, $|AB| = a$ and $|CD| = b$ then

$$\frac{1}{a^2 + \frac{1}{2}} < \frac{b}{a} < \frac{1}{a^2}.$$

2. The bisector of the angle B in the triangle ABC intersects the side AC at the point D. Let E be a point on side AB such that $3\angle ACE = 2\angle BCE$. The segments BD and CE intersect at the point P. One knows that $|ED| = |DC| = |CP|$. Find the angles of the triangle.

3. Let the sum of the two integers A and B be odd. Show that any integer can be written in the form $x^2 - y^2 + Ax + By$, where x and y are integers.

4. A and B are playing a game consisting of two parts:

• A and B make one throw each with a die. If the outcome is x and y, respectively, a list is created consisting of all two-digit integers $10a + b$, with a, $b \in \{1, 2, 3, 4, 5, 6\}$ such that $10a + b \leq 10x + y$.

For instance, if $x = 2$ and $y = 3$ the list is:

$$11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 21,\ 22,\ 23.$$

• The players now reduce the number of integers in the list by replacing a pair of the integers in the list by the non-negative difference of the chosen integers. If A, for instance, chooses 14 and 21 in the above example these two integers are removed and replaced by the integer 7. The new list becomes:

$$7,\ 11,\ 12,\ 13,\ 15,\ 16,\ 22,\ 23.$$

In the next move B may choose, for instance, 7 and 23, reducing the number of integers by one, and leaving the list

$$11,\ 12,\ 13,\ 15,\ 16,\ 16,\ 22.$$

The game is over when the list has been reduced to only one integer.

If the integer in the final list has the same parity as the outcome of A's throw, then A is the winner. What is the probability that A wins the game?

5. Let $s(m)$ denote the sum of the digits of the integer m. Prove that for any integer n, with $n > 1$ and $n \neq 10$, there is a unique integer $f(n) \geq 2$ such that $s(k) + s(f(n) - k) = n$ for all integers k satisfying $0 < k < f(n)$.

6. Let M be a set of real numbers. Assume that M is the union of a finite number of disjoint intervals and that the total length of the intervals is greater than 1. Prove that M contains at least one pair of distinct numbers whose difference is an integer.

As a second set collected by Chris Small, Canadian Team Leader, we give selected problems of the Ukrainian Mathematical Olympiad 1998.

UKRAINIAN MATHEMATICAL OLYMPIAD 1998
Selected Problems
April, 1998

1. (9th grade) Prove the inequality

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \;\geq\; 3$$

for positive real numbers a, b, c, with $abc = 1$.

2. (9th grade) A convex polygon with 2000 vertices in a plane is given. Prove that we may mark 1998 points of the plane so that any triangle with vertices which are vertices of the polygon has exactly one marked point as an internal point.

3. (10th grade) Let M be an internal point on the side AC of a triangle ABC, and let O be the intersection point of perpendiculars from the mid-points of AM and MC to lines BC and AB respectively. Find the location of M such that the length of segment OM is minimal.

4. (11th grade) A triangle ABC is given. Altitude CD intersects the bisector BK of ABC, and the altitude KL of BKC, at the points M and N respectively. The circumscribed circle of BKN intersects segment AB at the point $P \neq B$. Prove that triangle KPM is isosceles.

5. (11th grade) For real numbers x, y, $z \in (0,1]$, prove the inequality

$$\frac{x}{1+y+zx}+\frac{y}{1+z+xy}+\frac{z}{1+x+yz} \;\leq\; \frac{3}{x+y+z}\,.$$

6. (11th grade) The function $f(x)$ is defined on $[0,1]$ and has values in $[0,1]$. It is known that $\lambda \in (0,1)$ exists such that $f(\lambda) \neq 0$ and $f(\lambda) \neq \lambda$. Also,

$$f(f(x)+y) \;=\; f(x)+f(y)$$

for all x and y from the range of definition of the equality.

(a) Give an example of such a function.

(b) Prove that for any $x \in [0,1]$,

$$\underbrace{f(f(\ldots f}_{19}(x)\ldots)) \;=\; \underbrace{f(f(\ldots f}_{98}(x)\ldots))\,.$$

7. (11th grade) Two spheres with distinct radii are externally tangent at point P. Line segments AB and CD are given such that the first sphere touches them at the points A and C, and the second sphere touches them at the points B and D. Let M and N be the orthogonal projections of the mid-points of segments AC and BD on the line joining the centres of the given spheres. Prove that $PM = PN$.

8 (11th grade) Let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of real numbers such that

$$x_1 = 1, \quad x_{n+1} = \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2, \quad n \geq 1.$$

Prove that

(a) $x_{n+1} \geq x_n$ for all $n \geq 4$;

(b) $[x_n] = n$ for all $n \geq 4$ ($[a]$ denotes the integer part of a).

Next we give the problems of the Vietnamese Mathematical Olympiad, Category A, 1998. Thanks again go to Chris Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 1998

Category A, Day 1

March 13, 1998 — Time: 3 hours

1. Let $a \geq 1$ be a real number. Define a sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers by

$$x_1 = a, \; x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1 + \ln x_n}\right).$$

Prove that the sequence (x_n) has a finite limit, and determine it.

2. Let $ABCD$ be a tetrahedron and AA_1, BB_1, CC_1, DD_1 be diameters of the circumsphere of $ABCD$. Let A_0, B_0, C_0 and D_0 be the centroids of the triangles BCD, CDA, DAB and ABC, respectively. Prove that

(a) the lines A_0A_1, B_0B_1, C_0C_1 and D_0D_1 have a common point, which is denoted by F;

(b) the line passing through F and the mid-point of an edge is perpendicular to its opposite edge.

3. Let $\{a_n\}$ $(n = 0, 1, 2, \ldots)$ be a sequence of positive integers defined recursively by

$$a_0 = 20, \; a_1 = 100, \; a_{n+2} = 4a_{n+1} + 5a_n + 20.$$

Determine the smallest positive integer h for which $a_{n+h} - a_h$ is divisible by 1998 for every non-negative integer n.

Category A, Day 2

March 14, 1998 — Time: 3 hours

4. Prove that there does not exist an infinite sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers satisfying the two following conditions simultaneously

$$|x_n| \leq 0.666 \quad \text{for } n = 1, 2, \ldots, \tag{1}$$

$$|x_n - x_m| \geq \frac{1}{n(n+1)} + \frac{1}{m(m+1)} \tag{2}$$

for all $m \neq n$ $(m, n = 1, 2, \ldots)$.

5. Determine the smallest possible value of the following expression

$$F(x,y) = \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$$

where x, y are real numbers.

6. Determine all positive integers n for which there exists a polynomial $P(x)$ with real coefficients satisfying

$$P(x^{1998} - x^{-1998}) = x^n - x^{-n}$$

for all real numbers $x \neq 0$.

As a final set to get your solving skills going, we give the problems of Category B of the Vietnamese Mathematical Olympiad 1998. Again, thanks go to Chris Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 1998

Category B, Day 1

March 13, 1998 — Time: 3 hours

1. Let a be a real number. Define a sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers by

$$x_1 = a, \quad x_{n+1} = \frac{x_n(x_n^2 + 3)}{3x_n^2 + 1}$$

for $n \geq 1$. Prove that the sequence has a finite limit, and determine it.

2. Let P be a point lying on a given sphere. Three mutually perpendicular rays from P intersect the sphere at points A, B and C. Prove that for all such triads of rays from P, the plane of the triangle ABC passes through a fixed point, and determine the largest possible value of the area of the triangle ABC.

3. Let a, b be integers. Define a sequence $\{a_n\}$ $(n = 0, 1, 2, \dots)$ of integers defined by

$$a_0 = a\,,\, a_1 = b\,,\, a_2 = 2b - a + 2\,,\, a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$$

for $n \geq 0$.

(a) Find the general term of the sequence.

(b) Determine all integers a, b, for which a_n is a perfect square for all $n \geq 1998$.

Category B, Day 2

March 14, 1998 — Time: 3 hours

4. Let $x_1, x_2, \dots, x_n$ $(n \geq 2)$ be real positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}\,.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998\,.$$

5. Determine the smallest possible value of the following expression

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-3)^2}$$

where x, y are real numbers such that $2x - y = 2$.

6. Prove that for each positive odd integer n there is exactly one polynomial $P(x)$ of degree n with real coefficients satisfying

$$P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n}$$

for all real $x \neq 0$.

Determine if the above assertion holds for positive even integers n.

Now we give Klamkin's answers for the five Quickies given at the start of this number.

SOLUTIONS TO "AND FIVE MORE KLAMKIN QUICKIES"

1. Let the three lines be the x, y, and z–axes of a rectilinear coordinate system and the equation of the sphere be $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$. The required distance squared is $a^2 + b^2 + c^2$. Since the distance from the centre of the sphere to each of the lines is R, we have

$$R^2 \;=\; b^2+c^2 \;=\; c^2+a^2 \;=\; a^2+b^2\,.$$

Hence, $D^2 = 3R^2/2$.

2. By the Remainder Theorem,

$$P(x,y,z,t) \;=\; (x^2+y^2+z^2-t^2)Q(x,y,z,t)+R(x,y,z)t+S(x,y,z)$$

where Q, R and S are polynomials. Now, letting t be successively $\pm(x^2+y^2+z^2)^{1/2}$, it follows that $R = S = 0$.

3. Extend the chords QP and RP to intersect the circle again at points Q' and R'. It now follows that the arcs QR and $Q'R'$ are congruent and thus, their measures are $\pi - 2\theta$. Then if O is the centre, triangle OQR is isosceles whose vertex angle is also $\pi - 2\theta$. Hence, $QR = 2r\cos\theta$, which is the same for all P.

4. It follows from the given relation that both

$$\frac{(r(st))!}{(st)!(r!)^{st}} \quad \text{and} \quad \frac{(st)!}{t!(s!)^t}$$

are integers. Now just multiply them together.

5. Since

$$\begin{aligned}\frac{\sin y}{\sin(2x+y)} &= \frac{\sin(x+y)\cos x-\cos(x+y)\sin x}{\sin(x+y)\cos x+\cos(x+y)\sin x}\\ &= \frac{\tan(x+y)-\tan x}{\tan(x+y)+\tan x} \;=\; \sqrt{2}\,,\\ \text{so that}\quad \frac{\tan(x+y)}{\tan x} &= -3-2\sqrt{2} \;=\; \text{a constant.}\end{aligned}$$

Somehow we managed not to give the answers to the first five Klamkin Quickies given in the April Number of the *Corner*. Murray points out an error in the statement of problem 2.

SOLUTIONS TO FIVE KLAMKIN QUICKIES FROM APRIL 2001 CRUX with MAYHEM

1. Prove that

$$a+b+c \ \geq \ \sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\,,$$

where a, b, c are sides of a non-obtuse triangle.

Solution. By the power mean inequality

$$\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2} \ \leq \ 2c\,,$$

and similarly two other such inequalities. Then, adding, we get the desired result.

2. Determine the extreme values of the area of a triangle ABC, given the lengths of the two altitudes h_c, h_b. [Correction to question made.]

Solution. Let D and E be the feet of the altitudes h_b and h_c. Then by the Law of Sines applied to triangles ABD and ACE, $c = \frac{h_b}{\sin A}$ and $b = \frac{h_c}{\sin A}$. Twice the area is given by $2[ABC] = \frac{h_b h_c}{\sin A}$. Hence, the minimum area is $\frac{h_b h_c}{2}$ occuring for $A = \frac{\pi}{2}$. Also, by letting A approach π, the area becomes unbounded. In this case a would be arbitrarily large.

3. Determine the maximum area of a triangle ABC given the perimeter p and the angle A.

Solution. Since $2[ABC] = bc\sin A$, we have to maximize bc subject to

$$p \ = \ a+b+c, \quad \text{and} \quad a^2 \ = \ b^2+c^2-2bc\cos A\,.$$

Since $p-a = b+c \geq 2\sqrt{bc}$, bc will be a maximum when $b=c$ regardless of the value of a. Thus, we have

$$p \ = \ a+2b \quad \text{and} \quad a^2 \ = \ 2b^2-2b^2\cos A\,.$$

Then $(p-2b)^2 = 2b^2-2b^2\cos A$. Solving for b:

$$b \ = \ \frac{p\left(1+\sin\frac{A}{2}\right)}{2\cos^2\frac{A}{2}}$$

so that

$$\max[ABC] \ = \ \frac{p^2\left(\tan\frac{A}{2}\right)\left(1+\sin\frac{A}{2}\right)^2}{4\cos^2\frac{A}{2}}\,.$$

4. Determine the minimum value of

$$\sum\left[\frac{(a_2+a_3+a_4+a_5)}{a_1}\right]^{1/2}$$

where the sum is cyclic over the positive numbers a_1, a_2, a_3, a_4, a_5.

Solution. Applying the AM–GM Inequality to each term of the sum, the given sum is greater than or equal to

$$\sum 2\left(\frac{a_2a_3a_4a_5}{a_1^4}\right)^{1/8}$$

(where again the sum is cyclic). Finally applying the AM–GM Inequality again, the latter sum is greater than or equal to 10. There is equality in the given inequality if and only if the a_i's are equal. In a similar fashion it follows that if we increase the number of variables to $n+1$ and change the $1/2$ power to any positive number p, the minimum here would be $(n+1)^p$.

5. $ABCD$ and $AB'C'D'$ are any two parallelograms in a plane with A opposite to C and C'. Prove that BB', CC' and DD' are possible sides of a triangle.

Solution. Let the vectors from A to B and A to D be denoted by $\mathbf{U_1}$ and $\mathbf{U_2}$, and the vectors from A to B' and A to D' be denoted by $\mathbf{V_1}$ and $\mathbf{V_2}$. Then

$$\mathbf{BB'} = \mathbf{V_1}-\mathbf{U_1},\ \mathbf{DD'} = \mathbf{V_2}-\mathbf{U_2},\ \text{ and }\ \mathbf{CC'} = \mathbf{V_1}+\mathbf{V_2}-\mathbf{U_1}-\mathbf{U_2}\,,$$

so that $\mathbf{CC'} = \mathbf{BB'}+\mathbf{DD'}$. The rest follows from the triangle inequality $|\mathbf{P}\pm\mathbf{Q}| \le |\mathbf{P}|+|\mathbf{Q}|$ and with equality only if $\mathbf{P}$ and $\mathbf{Q}$ have the same direction.

Remarks: Christopher J. Bradley of Clifton College, Bristol, UK, also submitted solutions for the quickies, noting a problem with problem 2. He points out that he gave problem 1 to a group of students in 1988.

Next a comment answering an editorial question we posed.

1. [2000 : 325] Let m and n be natural numbers such that m^2+n^2 divides into mn. Prove that $m=n$.

Comment by Achilleas Sinefakopoulos, student, University of Athens, Greece.

This should be read "mn divides m^2+n^2". The same problem was posed in TOURNAMEN 18, Spring 1997. (See the solution, page 157 in "International Mathematics Tournament of the Towns, 1993–1997", edited by P.J. Taylor and A.M. Storozhev, Australian Mathematics Trust, 1998, Australia.

It seems that a package of solutions from long-time contributor Miguel Amengual Covas, Cala Figuera, Mallorca, Spain must have gone astray. Last issue we corrected the record for some. In this number, I want to acknowledge that he sent solutions to two problems for which we gave solutions in the April number of the *Corner*: 4. [1999 : 199–200; 2001 : 181] and 7. [1999 : 199–200; 2001 : 183] of the Ninth Irish Mathematical Olympiad.

Next we continue with readers' solutions to the St. Petersburg City Mathematical Olympiad, Third Round [1999 : 262] which we began last issue.

ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD
Third Round – February 25, 1996
11^{th} Grade (Time: 4 hours)

5. Find all quadruplets of polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ with real coefficients possessing the following remarkable property: for all integers x, y, z, t satisfying the condition $xy - zt = 1$, the equality $p_1(x)p_2(y) - p_3(z)p_4(t) = 1$ holds.

Solution by Pierre Bornsztein, Pontoise, France.

If p_1, p_2, p_3, p_4 are such polynomials, then for all $x \in \mathbb{Z}$:

$$p_1(x+1)p_2(1) - p_3(x)p_4(1) \;=\; 1 \tag{1}$$

and

$$p_1(x+1)p_2(1) - p_3(1)p_4(x) \;=\; 1\,. \tag{2}$$

Thus,

$$p_3(x)p_4(1) \;=\; p_3(1)p_4(x)\,. \tag{3}$$

Moreover,

$$p_1(x)p_2(1) - p_3(x-1)p_4(1) \;=\; 1$$

and

$$p_1(1)p_2(x) - p_3(x-1)p_4(1) \;=\; 1\,.$$

Thus,

$$p_1(x)p_2(1) \;=\; p_1(1)p_2(x)\,. \tag{4}$$

Moreover:

$$p_1(x)p_2(x) - p_3(x-1)p_4(x+1) \;=\; 1 \tag{5}$$

and

$$p_1(x^2)p_2(1) - p_3(x-1)p_4(x+1) \;=\; 1\,.$$

Thus,

$$p_1(x^2)p_2(1) \;=\; p_1(x)p_2(x)\,. \tag{6}$$

Since (3), (4), (5), (6) hold for an infinite number of values, and since they are relations between polynomials, then they hold for all real numbers x.

Case 1: If $p_1 \equiv 0$, then from (1):

$$p_3(x)p_4(1) = -1 \quad \text{for all } x \in \mathbb{R}\,.$$

It follows that p_3 is constant.

Since p_3 and p_4 are playing symmetric parts, we also have p_4 is constant. Then $p_3(x) = c$ and $p_4(x) = -\frac{1}{c}$ where $c \in \mathbb{R}^*$.

Conversely: $(0, p_2, c, -\frac{1}{c})$, where $p_2 \in \mathbb{R}[x]$ is arbitrary and $c \in \mathbb{R}^*$, is a solution of the problem.

By the same reasoning:

- $p_2 \equiv 0$ gives the solutions $(p_1, 0, c, -\frac{1}{c})$ where p_1 is an arbitrary polynomial, $p_1 \in \mathbb{R}[x]$, and $c \in \mathbb{R}^*$.
- $p_3 \equiv 0$ gives the solutions $(a, \frac{1}{a}, 0, p_4)$.
- $p_4 \equiv 0$ gives the solutions $(a, \frac{1}{a}, p_3, 0)$ where $a \in \mathbb{R}^*$ and p_3 (resp. p_4) is an arbitrary polynomial.

Now, we suppose that no one of p_1, p_2, p_3, p_4 is identically zero.

Case 2: If $p_1 \equiv a$ where $a \in \mathbb{R}^*$, then from (4), we have $p_2(x) = b$ where $b \in \mathbb{R}^*$.

If $p_3(1) = 0$, then, from (2) we have $ab = 1$. Thus, from (5), we have $p_3(x-1)p_4(x+1) = 0$ for all $x \in \mathbb{Z}$.

It follows that at least one of the polynomials p_3, p_4 has an infinite number of zeros. Thus, $p_3 \equiv 0$ or $p_4 \equiv 0$. Contradiction. We deduce that $p_3(1) \neq 0$.

Then, from (2), p_4 is constant: $p_4 \equiv d$, where $d \in \mathbb{R}^*$. In the same way, $p_3 \equiv c$, where $c \in \mathbb{R}^*$. And we must have $ab - cd = 1$.

Conversely: (a, b, c, d), where a, b, c, $d \in \mathbb{R}^*$ and $ab - cd = 1$, is a solution.

Notice that the condition a, b, c, d non-zero can be eased if $ab - cd = 1$ because it gives solutions of the first case.

Moreover, by the same reasoning, the cases p_3 is constant, p_4 is constant, p_2 is constant, give the same solutions.

Now, we suppose that:

Case 3: No one of p_1, p_2, p_3, p_4 is constant.

Then, for all $x \in \mathbb{Z}$: $p_1(x)p_2(0) - p_3(-1)p_4(1) = 1$ where p_1 is a non-constant polynomial. Thus, $p_2(0) = 0$.

In the same way, $p_1(0) = p_3(0) = p_4(0) = 0$. Then, we have $p_1(1)p_2(1) = p_1(1)p_2(1) - p_3(0)p_4(0) = 1$. It follows that $p_1(1) \neq 0$ and $p_2(1) \neq 0$.

In the same way, $-p_3(1)p_4(-1) = p_1(0)p_2(0) - p_3(1)p_4(-1) = 1$. Then $p_3(1) \neq 0$, $p_4(-1) \neq 0$.

But, it is clear that if (p_1, p_2, p_3, p_4) is a solution then $(ap_1, \frac{1}{a}p_2, bp_3, \frac{1}{b}p_4)$ is also a solution where a, b are arbitrary non-zero real numbers.

Then, with no loss of generality, we suppose that $p_1(1) = p_3(1) = 1$. Thus, $p_2(1) = 1$, $p_4(-1) = -1$.

From (4) and (6), we have $p_1(x^2) = p_1^2(x)$ for all $x \in \mathbb{R}$. We let $p_1(x) = \sum\limits_{i=0}^{n} a_i x^i$, $n \in \mathbb{N}^*$.

Identifying the coefficients, we have:

- $a_n = a_n^2$ with $a_n \neq 0$, then $a_n = 1$.
- $2a_n a_{n-1} = 0$, then $a_{n-1} = 0$.
- $2a_n a_{n-2} + a_{n-1}^2 = a_{n-1}$.

From the above, we have $a_{n-2} = 0$. And so on ... an easy induction leads to $a_k = 0$ for all $k < n$. Then $p_1(x) = x^n$ for some $n \in \mathbb{N}^*$.

From (4), we deduce that $p_2(x) = x^n$.

From (1), we have

$$(x+1)^n - 1 = p_3(x)p_4(1) = p_4(x)p_3(1) = p_4(x) .$$

Then,

$$p_4(x) = (x+1)^n - 1$$

and

$$p_3(x) = \frac{(x+1)^n - 1}{2^n - 1} \quad (\text{since } p_3(1) = 1) .$$

Then, for all x, y, z, t integers such that $xy - zt = 1$, we have

$$(xy)^n - \frac{((z+1)^n - 1)((t+1)^n - 1)}{2^n - 1} = 1 .$$

Thus, for all $t \in \mathbb{Z}$, with $z = -1$, $x = 1$, $y = 1 - t$, we must have

$$(1-t)^n + \frac{(1+t)^n - 1}{2^n - 1} = 1 .$$

It follows that the polynomial $Q(t) = (1-t)^n + \frac{(1+t)^n - 1}{2^n - 1}$ is constant. Since $n \geq 1$, the coefficient of t^n must be zero. That is,

$$(-1)^n + \frac{1}{2^n - 1} = 0 .$$

Then,

$$2^n - 1 = 1 \quad \text{or} \quad 2^n - 1 = -1 .$$

Thus, $n = 1$. It follows that

$$p_1(x) = p_2(x) = p_3(x) = p_4(x) = x .$$

Conversely: (x, x, x, x) is obviously a solution.

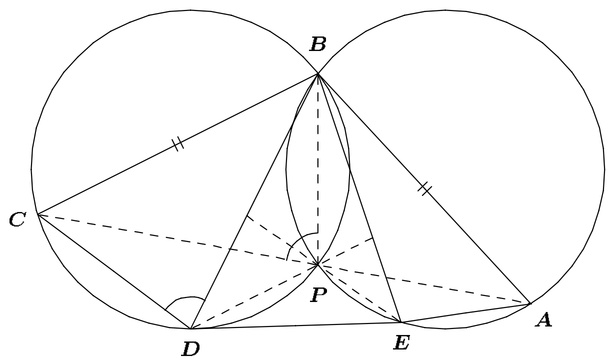
Conversely: The solutions are the quadruplets of the form:

(a) $\left(0, P, a, -\frac{1}{a}\right)$, $\left(P, 0, a, -\frac{1}{a}\right)$, $\left(a, \frac{1}{a}, P, 0\right)$, $\left(a, \frac{1}{a}, 0, P\right)$, where $P \in \mathbb{R}[x]$ and $a \in \mathbb{R}^*$ are arbitrary.

(b) (a, b, c, d), where a, b, c, $d \in \mathbb{R}$ with $ab - cd = 1$. This is, therefore, $\left(ax, \frac{1}{a}x, bx, \frac{1}{b}x\right)$ where a, $b \in \mathbb{R}^*$ are arbitrary.

6. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDE = 180^\circ$. Prove that the orthocentre of triangle BDE lies on diagonal AC.

Solution by Toshio Seimiya, Kawasaki, Japan.



(The condition $\angle AEB + \angle BDE = 180^\circ$ is incorrect. The correct condition is $\angle AEB + \angle BDC = 180^\circ$. After correcting, we shall solve the problem.)

Let P be the second intersection of the circumcircles of $\triangle BCD$ and $\triangle BAE$. Then

$$\angle APB = \angle AEB \quad \text{and} \quad \angle BPC = \angle BDC .$$

Therefore,

$$\angle APB + \angle BPC = \angle AEB + \angle BDC = 180^\circ .$$

Thus, A, P, C are collinear. That is, P is a point on the diagonal AC.

We put $\angle BAC = \angle BCA = \theta$ (because $AB = BC$). Then

$$\angle BEP = \angle BAP = \theta, \quad \text{and} \quad \angle BDP = \angle BCP = \theta .$$

Since $\angle ABE + \angle DBC = \angle EBD$, so that

$$\angle ABC = 2\angle EBD .$$

Since $\angle ABC + \angle BAC + \angle BCA = 180^\circ$, we have

$$2\angle EBD + 2\theta = 180^\circ .$$

Hence,

$$\angle EBD + \theta \;=\; 90^\circ\,.$$

Since $\angle EBD + \angle BDP = \angle EBD + \theta = 90^\circ$, and

$$\angle EBD + \angle BEP \;=\; \angle EBD + \theta \;=\; 90^\circ\,,$$

we have $DP \perp BE$ and $EP \perp BD$.

Hence, P is the orthocentre of $\triangle BDE$.

Thus, the orthocentre of $\triangle BDE$ lies on diagonal AC.

Now we turn to solutions to problems of the Selective Round, 11^{th} Grade of the St. Petersburg City Mathematical Olympiad [1999 : 263].

Selective Round – March 10, 1996

11^{th} Grade (Time: 5 hours)

1. It is known about real numbers $a_1, \ldots, a_{n+1}$; $b_1, \ldots, b_n$ that $0 \le b_k \le 1$ $(k = 1, \ldots, n)$ and $a_1 \ge a_2 \ge \cdots \ge a_{n+1} = 0$. Prove the inequality:

$$\sum_{i=1}^{n} a_k b_k \le \sum_{k=1}^{[\sum_{j=1}^{n} b_j]+1} a_k\,.$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's write-up.

Let $m = \left[\sum\limits_{j=1}^{n} b_j\right]$. If $m = 0$, then $0 \le \sum\limits_{j=1}^{n} b_j < 1$, so that

$$\sum_{k=1}^{n} a_k b_k \;\le\; a_1 \sum_{j=1}^{n} b_j \;\le\; a_1 \;=\; \sum_{k=1}^{m+1} a_k\,.$$

If $m = n$, then $b_k = 1$ $(k = 1, \ldots, n)$ and the inequality to be proved is obvious. Also, the case $n = 1$ is immediate, so that, in the following, we will suppose $n \ge 2$ and $1 \le m < n$.

Let $s(1)$ be the first integer ≥ 2 such that $\sum\limits_{j=1}^{s(1)} b_j \ge 1$,

$s(2)$ be the first integer $> s(1)$ such that $\sum\limits_{j=1}^{s(2)} b_j \ge 2$,

....

$s(m)$ be the first integer $> s(m-1)$ such that $\sum\limits_{j=1}^{s(m)} b_j \ge m$.

Note that $s(k) \geq k+1$, so that $a_{s(k)} \leq a_{k+1}$ $(k = 1, \ldots, m)$. Now,

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} c_k \text{ where } c_k = (a_k - a_{k+1}) \sum_{j=1}^{k} b_j \quad (\text{using } a_{n+1} = 0) \\ &= A_0 + A_1 + \cdots + A_m\end{aligned}$$

with

$$A_0 = \sum_{k=1}^{s(1)-1} c_k,\ A_1 = \sum_{k=s(1)}^{s(2)-1} c_k,\ \ldots,\ A_m = \sum_{k=s(m)}^{n} c_k\,.$$

Let $t \in \{0, 1, \ldots, m\}$. Then for $s(t) \leq k \leq s(t+1) - 1$ (defining $s(0) = 1$, $s(m+1) = n+1$), we have $\sum\limits_{j=1}^{k} b_j \leq t+1$ so that

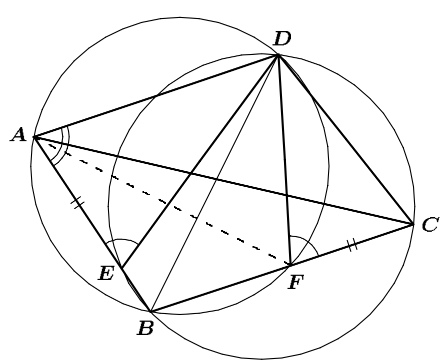
$$A_t \leq (t+1) \sum_{k=s(t)}^{s(t+1)-1} c_k = (t+1)(a_{s(t)} - a_{s(t+1)})\,.$$

Thus, $A_0 \leq a_1 - a_{s(1)}$, $A_1 \leq 2(a_{s(1)} - a_{s(2)})$, $A_2 \leq 3(a_{s(2)} - a_{s(3)})$, $\ldots$, $A_m \leq (m+1)(a_{s(m)} - a_{n+1}) = (m+1)a_{s(m)}$, and, adding:

$$\sum_{k=1}^{n} a_k b_k \leq a_1 + a_{s(1)} + \cdots + a_{s(m)} \leq a_1 + a_2 + \cdots + a_{m+1}\,.$$

2. Segments AE and CF of equal length are taken on the sides AB and BC of a triangle ABC. The circle going through the points B, C, E and the circle going through the points A, B, F intersect at points B and D. Prove that the line BD is the bisector of angle ABC.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Since A, B, F, D are concyclic, we have

$$\angle EAD = \angle BAD = \angle CFD. \tag{1}$$

Similarly, we have

$$\angle AED = \angle FCD. \tag{2}$$

Since $AE = CF$, we get from (1) and (2) that $\triangle DAE \equiv \triangle DFC$. Therefore, $DA = DF$. Hence, we have

$$\angle ABD = \angle AFD = \angle FAD = \angle FBD = \angle CBD.$$

Thus, BD is the bisector of $\angle ABC$.

3. Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap+bq$ is also prime.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We use a solution derived from the similar solutions of Aassila and Evagelopoulos.

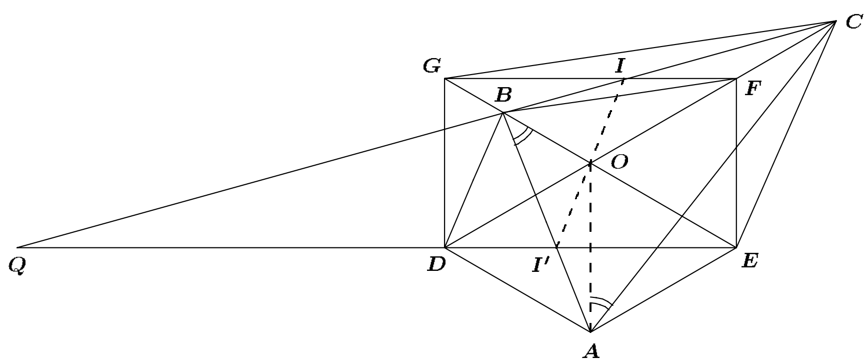
We are going to use the *reductio ad absurdum* method in order to prove the hypothesis of the problem.

Suppose there are positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

By Dirichlet's Theorem, there exist infinitely many primes in any non-zero residue class modulo m; this means there exists a pair p, q such that $p \equiv b \,(\text{mod}\ m)$, $q \equiv -a \,(\text{mod}\ m)$, giving $ap + bq$ is divisible by m, a contradiction.

5. In a triangle ABC the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. A point D is chosen on the half-line CO such that the triangle AOD is equilateral. The mid-perpendicular to the segment AO meets the line BC at point Q. Prove that the line OQ divides the segment BD into two equal parts.

Solutions by René Bornsztein, Antony, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of René Bornsztein.



We have $\angle BAC = 60^\circ$, and

$$\angle BOC = \angle AOB = \angle AOC = 120^\circ .$$

Then

$$\angle OBA = 60^\circ - \angle BAO = \angle BAC - \angle BAO = \angle OAC .$$

Thus, $\triangle OAB$ and $\triangle OCA$ are similar. It follows that

$$\frac{OB}{OA} = \frac{OA}{OC} . \qquad (1)$$

Let E be the point such that $OEAD$ is a diamond. Let G, F be the points such that $EDGF$ is a rectangle, whose diagonals intersect with an angle of 120°.

Then $G \in (OE)$, $F \in (OD)$, and $B \in (GE)$.

It follows that:

$$\frac{OB}{OE} = \frac{OB}{OA} = \frac{OA}{OC} = \frac{OD}{OC} \quad \text{(from (1).)}$$

Then (BD) is parallel to (EC).

Moreover,

$$\frac{OB}{OG} = \frac{OB}{OA} = \frac{OA}{OC} = \frac{OF}{OC} .$$

Then (BF) is parallel to (GC).

Let I, I' be respectively the intersection of (BC) and (GF), of (OI) and (ED). Since O is the centre of $DEFG$, O is the mid-point of (II').

From Thales' Theorem,

- In IGC and IFB: $\dfrac{BF}{GC} = \dfrac{BI}{IC}$.
- In OBF and OGC: $\dfrac{BF}{GC} = \dfrac{OB}{OG} = \dfrac{OB}{OE}$.

It follows that (OI) is parallel to (EC), and then to (BD).

Since Q, C, I, B are collinear, and Q, D, I' are collinear, since (OQ) divides (II') into two equal parts, then (OQ) divides (BD) into two equal parts.

Remark: (OQ) also divides (CE) into two equal parts.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations as well as Olympiad Contests!

BOOK REVIEWS

ALAN LAW

Geometry at Work,
edited by Catherine A. Gorini,
published by the Mathematical Association of America (MAA Notes, No. 53),
2000, ISBN 0-88385-164-4, softcover, 300 pages, $25.95 (U.S.).
Reviewed by **J. Chris Fisher**, *University of Regina, Regina, Saskatchewan.*.

Geometry at Work is a collection of twenty essays in applied geometry, intended for "anyone having a college-level course in geometry." The preface states that the articles can be used as "supplementary materials for teachers, resources for student projects, ideas for special lectures, inspiration for further research, or simply to broaden one's awareness of geometry and its applications." They are arranged into six sections:

Art and Architecture (4 papers, 34 pages),

Vedic Civilization (2 papers, 17 pages),

The Classroom (teaching descriptive geometry and ethnomathematics; 2 papers, 10 pages),

Engineering (robotics, structural engineering, Geographical Information Systems, and medical imaging; 5 papers, 47 pages),

Decision-Making Processes (voting and computer learning; 2 papers, 25 pages),

Mathematics and Science (number theory, optimization, graph theory, quantum mechanics, and crystallography; 5 papers, 69 pages).

The level is that of the various journals of the *Mathematical Association of America*, although the mathematical content is variable — there were several articles with insufficient content for publication in any mathematical journal, and one whose level was well beyond the intended audience. This exception is "Three-Dimensional Topology and Quantum Physics" by Louis H. Kauffman. It is a marvelous survey of quantum mechanics from the beginning in 1924 with DeBroglie's "fantastic notion" that inspired Schrodinger's equations, through the recent advances in knot theory. Although one frequently sees hints that the pieces are connected, I have never before seen the explanation of how the pieces fit together. The story is fascinating, informative, very well motivated, and told in just 11 pages. The geometry involved is the intuitive notion of topology behind Reidemeister's basic ideas about knots, but there are many advanced concepts — eigenvalues of Hilbert-space operators, for one — that are an essential part of the development.

Before continuing my review, here are a few very personal opinions. The motivation behind collecting applications seems based on the notion that geometry is a collection of usable facts. I suppose so, to some extent, but this point of view unfortunately reinforces the beliefs of those responsible for cutting back the teaching of geometry in high schools across North America. In many schools geometry is taught as a bunch of facts that can be stuck here and there as topics in a general mathematics course. However, geometry used to be, and always should be, the first place where students get to see deductive reasoning. The intuitive nature of the subject helps a person develop a feeling for the role of proof in mathematics and science. The abstract axiom systems studied in university courses do not serve as an effective pedagogical tool. For a full discussion of this "application" of geometry I recommend the article by Jim McClure, "Start Where They Are: Geometry as an Introduction to Proof," *Amer. Math. Monthly* **107**:1 (January 2000) 44–52.

The articles are all well written. Basic geometry is represented in several articles in its historic role of guiding measurements both in traditional settings and in medical imaging. Somewhat more advanced geometry is needed to analyze aperiodic tilings, where symmetry is replaced by statistical symmetry. Convexity plays the key role in several articles, of which two involve optimization, one involves number theory (applying Minkowski's geometry of numbers to prove three familiar results), and one deals with robotics: how many fingers are needed to "grasp" a rigid 3–dimensional object? (Answer: 7 or so, depending on what is meant by *grasp*.)

The real question: is it worthwhile to purchase the book? My opinion is neutral here. The answer depends on how much money and shelf space is available to you or to your library. Eleven of the essays are of questionable value: either they lack mathematical content, or similar articles are readily available elsewhere, or they are too sketchy. (One of these sketchy articles looked promising, but its three references were all in Russian.) That leaves nine articles that I found worth reading carefully. Any reader of ***CRUX with MAYHEM*** would likewise find several interesting articles in this collection. Had somebody asked my opinion, I would have suggested that these articles appear in a widely accessible refereed journal. Far more valuable than this monograph would have been a bibliography that provides a brief description of articles involving applications and classifies them according to the mathematics required and the level of the exposition.

An Interesting Arithmetic Problem

Jingcheng Tong (Professor), Edith Atkins (Student)
and Debra Simonson (Student)

In the textbook [1], one can find the following problem:

Let A, B, C, a, b, c, α, β, γ be distinct digits from 1, 2, 3, 4, 5, 6, 7, 8, 9. Give an example such that the equality below holds.

$$\begin{array}{r} A\,a\,\alpha \\ +\,B\,b\,\beta \\ \hline C\,c\,\gamma \end{array}$$

Students turned in many different results. Some of them gave even 20 solutions. The record in our class of thirty students is made by the second author of this article, who found 120 such examples.

Thus, the question is: How many such examples are there?

It is natural to check all possible choices of A, B, a, b, α, β from the nine digits 1, 2, $\dots$, 9, then add them together to find the solutions. That is to say, let A be one of the 9 digits, B be one of the remaining 8 digits, a be one of the remaining 7 digits, and so on. Then there are $9{\cdot}8{\cdot}7{\cdot}6{\cdot}5{\cdot}4 = 60480$ possible choices for checking. It is not reasonable to check by hand such a large amount of operations.

We will solve this problem by using some most basic counting techniques which could be useful for school teachers. The strategy to solve this complicated problem is to divide it into many small easier problems.

(1) Let us analyze one example to gain insight into the solution of the problem as instructed by George Pólya in [2].

$$\begin{array}{r} 2\ 3\ 5 \\ +\,7\ 4\ 6 \\ \hline 9\ 8\ 1 \end{array}$$

If we switch 2, 7, or 3, 4, or 5, 6 in the above example, we still have equality. Therefore, if one example is found, there are indeed $2^3 = 8$ examples. We will use the example with $A < B$, $a < b$, $\alpha < \beta$ as the representative of the group of eight examples.

(2) $A+B+C+a+b+c+\alpha+\beta+\gamma = 1+2+3+4+5+6+7+8+9 = 45$. Hence,

$$A + B + a + b + \alpha + \beta \;=\; 45 - (C + c + \gamma)\,.$$

(3) It is impossible that, in our problem, $A + B = C$, $a + b = c$ and $\alpha + \beta = \gamma$ hold simultaneously. For if so, we then have

$$\begin{aligned} A + B + a + b + \alpha + \beta &= C + c + \gamma\,, \\ 45 - (C + c + \gamma) &= C + c + \gamma\,, \\ C + c + \gamma &= \frac{45}{2}\,. \end{aligned}$$

This is a contradiction since the sum of three digits must be a positive integer.

(4) It is impossible that in our problem $A + B = C$, $a + b = 10 + c$ and $\alpha + \beta = 10 + \gamma$ hold simultaneously. For if so, we then have

$$\begin{aligned} A + B + a + b + \alpha + \beta &= 20 + (C + c + \gamma)\,, \\ 45 - (C + c + \gamma) &= 20 + (C + c + \gamma)\,, \\ C + c + \gamma &= \frac{25}{2}\,. \end{aligned}$$

This is a contradiction.

(5) Either (i) $A + B = C$, $a + b = c$ and $\alpha + \beta = 10 + \gamma$ or (ii) $A + B = C$, $a + b = 10 + c$ and $\alpha + \beta = \gamma$ holds. In each case, we prove that $C + c + \gamma = 18$.

Take, for instance, case (i): since $\alpha + \beta = 10 + \gamma$, when practicing the addition we have to add 1 unit to the second column, which we denote as follows

$$\begin{array}{r} A\,a\;\alpha \\ +\,B\,b.\,\beta \\ \hline C\,c\;\gamma \end{array}$$

Thus,

$$A + B = C\,, \qquad a + b + 1 = c\,, \qquad \alpha + \beta = 10 + \gamma\,.$$

Hence,

$$\begin{aligned} A + B + a + b + 1 + \alpha + \beta &= 10 + (C + c + \gamma)\,, \\ 45 - (C + c + \gamma) + 1 &= 10 + (C + c + \gamma)\,, \\ C + c + \gamma &= 18\,. \end{aligned}$$

Case (ii) can be similarly discussed.

(6) From the discussion (5), we know that, if an example is found, there is another example obtained by switching the order of the second and third columns. In the following expression, if the left is a solution, the right one is automatically a solution.

Since there is only one carry digit, as long as the column consisting of α, β, γ and the carry bit column stay together, the "summands" can be rearranged.

$$\begin{array}{r} A\,a\ \alpha \\ +\,B\,b.\,\beta \\ \hline C\,c\ \gamma \end{array} \qquad\qquad \begin{array}{r} a\ \alpha\ A \\ +\,b.\,\beta\ B \\ \hline c\ \gamma\ C \end{array}$$

For simplicity, we pick up the example where the third column satisfies $\alpha+\beta = 10+\gamma$ as the representative. Combining with the discussion in (1), we may use one representative to represent a group of $8 \times 2 = 16$ examples.

(7) Now we discuss one possibility for $C + c + \gamma = 18$.

$1+8+9 = 18$. The remaining digits are $\{2, 3, 4, 5, 6, 7\}$. The possible choice of α, β from this remaining set of digits must be either (i) $4 + 7 = 11$ or (ii) $5 + 6 = 11$.

In case (i), since $2 + 6 = 8$, $3 + 5 = 8$ and $3 + 6 = 9$, we have three representative examples:

$$\begin{aligned} 234 + 657 &= 891\,, \\ 324 + 567 &= 891\,, \\ 324 + 657 &= 981\,. \end{aligned}$$

In case (ii), since $2 + 7 = 9$, we have one more representative example:

$$235 + 746 = 981\,.$$

(8) Other cases for $C + c + \gamma = 18$ are discussed as follows:

$2 + 7 + 9$ with remaining digits $\{1, 3, 4, 5, 6, 8\}$, possible choice of α, β is $4 + 8 = 12$, there are three representative examples:

$$\begin{aligned} 134 + 658 &= 792\,, \\ 214 + 758 &= 972\,, \\ 314 + 658 &= 972\,. \end{aligned}$$

$3 + 6 + 9$ with remaining digits $\{1, 2, 4, 5, 7, 8\}$, possible choice of α, β is $5 + 8 = 13$, there are two representative examples:

$$\begin{aligned} 215 + 478 &= 693\,, \\ 215 + 748 &= 963\,. \end{aligned}$$

$3 + 7 + 8$ with remaining digits $\{1, 2, 4, 5, 6, 9\}$, possible choice of α, β is $4 + 9 = 13$, there are three representative examples:

$$\begin{aligned} 124 + 659 &= 783\,, \\ 214 + 569 &= 783\,, \\ 214 + 659 &= 873\,. \end{aligned}$$

$4 + 5 + 9$ with remaining digits $\{1, 2, 3, 6, 7, 8\}$, since the last digit of the sum $\alpha + \beta$ must be either 4 or 5, the possible choices of α, β are (i) $6 + 8 = 14$, or (ii) $7 + 8 = 15$.

Case (i) gives two representative examples:

$$\begin{aligned} 216 + 378 &= 594\,, \\ 216 + 738 &= 954\,. \end{aligned}$$

Case (ii) gives two representative examples:

$$\begin{aligned} 127 + 368 &= 495\,, \\ 317 + 628 &= 945\,. \end{aligned}$$

$4 + 6 + 8$ with remaining digits $\{1, 2, 3, 5, 7, 9\}$, since the last digit of the sum $\alpha + \beta$ must be either 4 or 6, the possible choices of α, β are (i) $5 + 9 = 14$, or $7 + 9 = 16$.

Case (i) gives one representative example:

$$125 + 739 = 864\,.$$

Case (ii) gives two representative examples:

$$\begin{aligned} 127 + 359 &= 486\,, \\ 317 + 529 &= 846\,. \end{aligned}$$

$5 + 6 + 7$ with remaining digits $\{1, 2, 3, 4, 8, 9\}$, possible choice of α, β is $8 + 9 = 17$, there are two representative examples:

$$\begin{aligned} 128 + 439 &= 567\,, \\ 218 + 439 &= 657\,. \end{aligned}$$

(9) Conclusion: From the discussion of (7) and (8), there are 21 representatives. Therefore, there are altogether $16 \times 21 = 336$ examples.

The beauty of mathematics exists everywhere, even in simple arithmetic.

Acknowledgement. The authors thank the referee sincerely for suggestions improving this paper.

References

1. Thomas Sonnabend (1997), *Mathematics for Elementary Teachers*, second edition, Saunders College Publishing.
2. R. Billstein, S. Libeskind and J. Lott (1996), *Mathematics for Elementary Teachers*, sixth edition, Addison-Wesley.

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THE SKOLIAD CORNER

No. 55

Shawn Godin

Welcome to the Skoliad Corner. I would like to thank Robert Woodrow for his work over the years since he started Skoliad. I hope that I can do justice to his creation.

The format of Skoliad will remain very much the same; the only difference will be the time lag between the contests and solutions. We would like to print solutions from readers, in particular, students in high school or elementary school. Please send any solutions to

The Skoliad Corner, c/o Shawn Godin
Cairine Wilson S.S. 975 Orleans Blvd
Gloucester, ON Canada K1C 2Z5

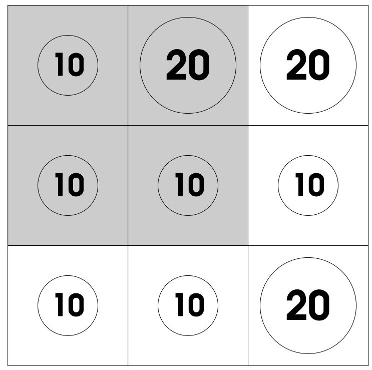
Please send your solutions to the problems in this issue by *1 January 2002*. Look for prizes for solutions in the new year.

The first entry to the corner is the 2000 National Bank Junior Mathematics Competition. The contest is written by students in years 9 to 11 and is organized by the University of Otago in Dunedin, New Zealand. My thanks go out to Warren Palmer and Derek Holton for forwarding the contest material to me. If you would like information on the contest feel free to contact them at

`nbjmc@maths.otago.ac.nz`

2000 National Bank Junior Mathematics Competition

1. In this problem we will be placing various arrangements of 10¢ and 20¢ coins on the nine squares of a 3×3 grid. Exactly one coin will be placed in each of the nine squares. The grid has four 2×2 subsquares each containing a corner, the centre, and the two squares adjacent to these. One example is shown in the diagram.



A 3×3 *grid with the top left subsquare shaded in. This subsquare contains a total of* 50¢*, while the others contain* 60¢*,* 40¢*, and* 50¢ *respectively.*

(a) Find an arrangement where the totals of the four 2×2 subsquares are 40¢, 60¢, 60¢ and 70¢ in any order.

(b) Find an arrangement where the totals of the four 2×2 subsquares are 50¢, 60¢, 70¢ and 80¢ in any order.

For each part of the problem below illustrate your answer with a suitable arrangement and an explanation of why no other suitable arrangement contains a larger (part(c)) or a smaller (part(d)) amount of money.

(c) What is the maximum amount of money which can be placed on the grid so that each of the 2×2 subsquares contains exactly 50¢?

(d) What is the minimum amount of money which can be placed on the grid so that the average amount of money in each of the 2×2 subsquares is exactly 60¢?

2. *Note: In this question an "equal division" is one where the total* **weight** *of the two parts is the same.*

(a) Belinda and Charles are burglars. Among the loot from their latest caper is a set of 12 gold weights of 1g, 2g, 3g, and so on, through to 12g. Can they divide the weights equally between them? If so, explain how they can do it; if not, why not?

(b) When Belinda and Charles take the remainder of the loot to Freddy the fence, he demands the 12g weight as his payment. Can Belinda and Charles divide the remaining 11 weights equally between them? If so, explain how they can do it; if not, why not?

(c) Belinda and Charles also have a set of 150 silver weights of 1g, 2g, 3g, and so on, through to 150g. Can they divide these weights equally between them? If so, explain how they can do it; if not, why not?

3. Humankind was recently contacted by three alien races: the Kweens, the Ozdaks, and the Merkuns. Little is known about these races except:

- Kweens always speak the truth.
- Ozdaks always lie.
- In any group of aliens a Merkun will never speak first. When it does speak, it tells the truth if the previous statement was a lie, and lies if the previous statement was truthful.

Although the aliens can readily tell one another apart, of course to humans all aliens look the same.

A high-level delegation of three aliens has been sent to Earth to negotiate our fate. Among them is at least one Kween. On arrival they make the following statements (in order):

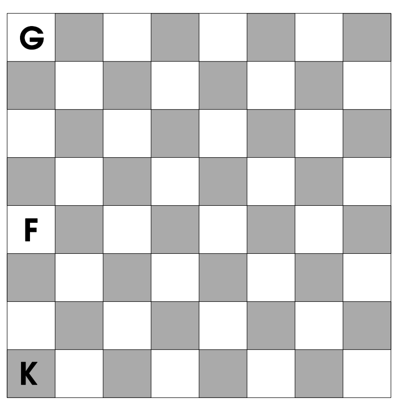
Statement A (First Alien): The second alien is a Merkun.

Statement B (Second Alien): The third alien is not a Merkun.

Statement C (Third Alien): The first alien is a Merkun.

Which alien or aliens can you be certain are Kween?

4. A chessboard is an 8×8 grid of squares. One of the chess pieces, the king, moves one square at a time in any direction, including diagonally.

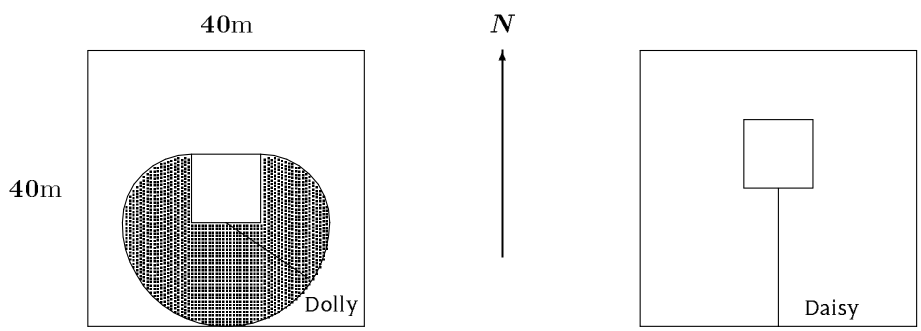


(a) A king stands on the lower left corner of a chessboard (marked **K**). It has to reach the square marked **F** in exactly 3 moves. Show that the king can do this in exactly **four** different ways.

(b) Assume that the king is placed back on the bottom left corner. In how many ways can it reach the upper left corner (marked **G**) in exactly **seven** moves?

5. *Note: For this question answers containing expressions such as $\frac{4\pi}{13}$ are acceptable.*

(a) The Jones family lives in a perfectly square house, 10m by 10m, which is placed exactly in the middle of a 40m by 40m section, entirely covered (except for the house) in grass. They keep the family pet, Dolly the sheep, tethered to the middle of one side of the house on a 15m rope. What is the area of the part of the lawn (in m^2) in which Dolly is able to graze? (See shaded area.)



The Jones   The Smiths

(b) The Jones' neighbours, the Smiths, have an identical section to the Jones but their house is located five metres to the North of the centre. Their pet sheep, Daisy, is tethered to the middle of the southern side of the house on a 20m rope. What is the area of the part of the lawn (in m^2) in which Daisy is able to graze?

Next we have part A of the final round of the BC senior mathematics competition. My thanks go to Jim Totten of the University College of the Cariboo, and Clint Lee of Okanagan University College for forwarding the material to me.

British Columbia Colleges
Senior High School Mathematics Contest, 2001
Final Round — Part A, Friday 4 May 2001

1. The number 2001 can be written as a difference of squares, $x^2 - y^2$, where x and y are positive integers, in four distinct ways. The sum of the four possible x values is:

(a) 55   (b) 56   (c) 879   (d) 1440   (e) 2880

2. Antonino goes to the local fruit stand and spends a total of $20.01 on peaches and pears. If pears cost $18\,¢$ and peaches cost $33\,¢$, the maximum number of fruits Antonino could have bought is:

(a) 110 (b) 107 (c) 100 (d) 92 (e) 62

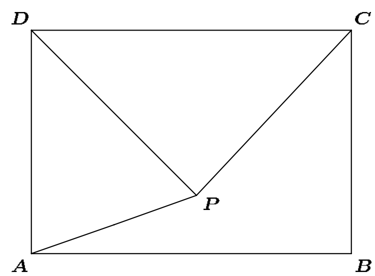
3. The value of $\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}$ is:

(a) 1 (b) 2 (c) $\sqrt{3}$ (d) $\sqrt{6}$ (e) 4

4. The point P is interior to the rectangle $ABCD$ such that $\overline{PA} = 3$ cm, $\overline{PC} = 5$ cm, and $\overline{PD} = 4$ cm. Then $\overline{PB}$, in centimetres, is:

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) $3\sqrt{3}$

(d) $4\sqrt{2}$ (e) 2



5. Two overlapping spherical soap bubbles, whose centres are 50 mm apart, have radii of 40 mm and 30 mm. The two spheres intersect in a circle whose diameter, in millimetres, is:

(a) 36 (b) 48 (c) 50 (d) 54 (e) 64

6. The local theatre charges one dollar for the Sunday afternoon matinée. One Sunday the cashier finds that he has no change. Eight people arrive at the theatre; four have only a one-dollar coin (a loonie) and four have only a two-dollar coin (a toonie). Depending on how the people line up, the cashier may or may not be able to make change for every person in the line as they buy their tickets one at a time. Suppose that the eight people form a line in random order, without knowing who has a loonie and who has a toonie. Then the probability that the cashier will be able to make change for every person in the line is:

(a) $\frac{1}{70}$ (b) $\frac{1}{14}$ (c) $\frac{1}{7}$ (d) $\frac{1}{5}$ (e) $\frac{1}{4}$

7. There is a job opening at bcmath.com for a Webmaster. There are three required skills for the position: Writing, Design, and Programming. There are 45 applicants for the position. Of the 45 applicants, 80% have at least one of the required skills. Twenty of the applicants have at least design skills, 25 have at least writing skills, and 21 have at least programming skills. Twelve of the applicants have at least writing and design skills, fourteen have at least writing and programming skills, and eleven have at least design and programming skills. If only those applicants with all three skills will be interviewed, the number of applicants to be interviewed is:

(a) 3 (b) 7 (c) 8 (d) 9 (e) 11

8. Let $a \textcircled{L} b$ represent the operation on two numbers a and b, which selects the larger of the two numbers, with $a \textcircled{L} a = a$. Let $a \textcircled{S} b$ represent the operation which selects the smaller of the two numbers with $a \textcircled{S} a = a$. If a, b, and c are distinct numbers, and $a \textcircled{S} (b \textcircled{S} c) = (a \textcircled{S} b) \textcircled{L} (a \textcircled{S} c)$, then we must have:

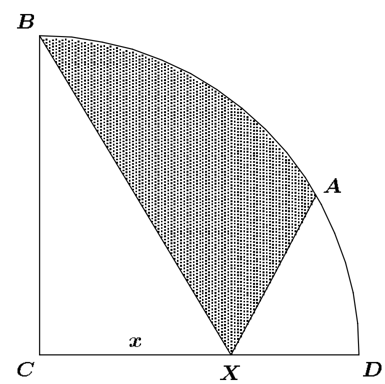
(a) $a < b$ and $a < c$ (b) $a > b$ and $a > c$ (c) $c < b < a$
(d) $c < a < b$ (e) $a < b < c$

9. The coordinates of the points A, B, and C are $(7, 4)$, $(3, 1)$, and $(0, k)$, respectively. The minimum value of $\overline{AC} + \overline{BC}$ is obtained when k equals:

(a) 1 (b) 1.7 (c) 1.9 (d) 2.5 (e) 4

10. Given the quarter circle BAD with radius $\overline{BC} = \overline{DC} = 1$, suppose that $\angle BCA = 60°$ and X is a point on segment DC with $\overline{CX} = x$. If the area of the shaded region BXA is one half the area of the quarter circle, then the value of x is:

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{\pi}{6}$
(d) $\frac{\pi}{4}$ (e) none of these



That completes the Skoliad Corner for this number. Send me your comments, suggestions, material and solutions for use in the corner.

Can you help?

One of our regular contributers, Dr. Eckard Specht <specht@iep352.nat.uni-magdeburg.de> has been looking for a very long time for two well-known books by Roger Arthur Johnson:

1. Advanced Euclidean Geometry, Dover Publications, Mineola, NY, 1960
2. Modern Geometry - An Elementary Treatise on the Geometry of the Triangle and the Circle, Houghton Mifflin, 1929.

Both are rare books and for a long time out of print. Attempts to order them by rare books shops (www.frugalfinder.com et al) on the internet failed. These books do not seem to exist in Germany — all libraries are missing these.

Does somebody have a copy of one or both that they would be willing to give or sell? If so, please contact Dr. Specht directly. Any help in obtaining a copy is appreciated. Thanks.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is

NEW! mayhem-editors@cms.math.ca NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Mayhem Problems

The Mayhem Problems editors are:

Chris Cappadocia	*Mayhem Problems Editor,*
Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 5 of 2002.

Mayhem Problems — NEW!

More changes! We have decided to make the problem section of MAYHEM more like that in CRUX. So, starting with this issue, **each** issue will contain problems for you to solve and, eventually, solutions. For the next few issues we will print solutions to outstanding problems from past High School, Advanced and Challenge sections. Sometime in the new year we should be totally in our new format.

To help with the changes to MAYHEM, the editors have secured a grant from the CMS endowment fund. We will be using this money to promote MAYHEM, as well as making 2002 a year of prizes for MAYHEM! Send in problems and solutions to proposed problems and you may win a prize.

There will be random prizes drawn from names of solvers and proposers as well as prizes for "problem solvers of the year". Watch for details in future problem sections.

Mayhem assistant editor Chris Cappadocia will be taking over the new Mayhem Problems section. We are still in the process of setting up a permanent address that problems and solutions can be sent to, so in the interim correspondence can be sent to the MAYHEM editor.

I would like to take a moment to thank Adrian Chan, Donny Cheung and David Savitt for their hard work on the MAYHEM problem sections over the years. We will still see their work for a bit as we round out the material from their columns, and we may see them pop up in some other capacity in the future. Thanks again guys, your effort is greatly appreciated!

To start the new problem section, we will begin with seven questions from various contests put on by the Australian Mathematics Trust. The questions were, originally, multiple choice but the choices have been removed (and possibly a word changed to get you to find, rather than choose) so that you have to do all the work yourself!

My thanks go to Peter Taylor, Executive Director of the Australian Mathematics Trust for permitting us to use the questions. For more information on the Trust, its contests and publications, you can visit its website `http://www.amt.canberra.edu.au` or email the Executive Director at

`pjt@amt.canberra.edu.au`

To facilitate their consideration, please send your proposals and solutions on signed and separate sheets of paper. These may be typewritten or neatly handwritten and should be received no later than 1 January 2002. They may also be sent by email (it would be appreciated if it was in LaTeX). Solutions received after this date will also be considered if there is sufficient time before the date of publication.

Australian Mathematics Trust Questions

M1. Four singers take part in a musical round of 4 equal lines, each finishing after singing the round through three times. The second singer begins when the first singer begins the second line,the third singer begins when the first singer begins the third line,the fourth singer begins when the first singer begins the fourth line. Find the fraction of the total singing time that all four are singing at the same time.

M2. When 5 new classrooms were built for Wingecarribee School the average class size was reduced by 6. When another 5 classrooms were built, the average class size reduced by another 4. If the number of students remained the same throughout the changes, how many students were there at the school?

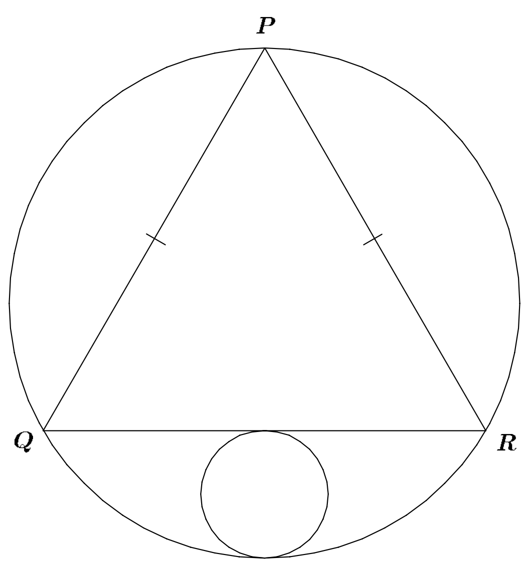
M3. How many years in the 21^{st} century will have the property that, dividing their year number by each of 2, 3, 5, and 7 always leaves a remainder of 1?

M4. We write down all the numbers $2, 3, \ldots, 100$, together with all their products taken two at a time, their products taken three at a time, and so on up to and including the product of all 99 of them. Find the sum of the reciprocals of all the numbers written down.

M5. The ratio of the speeds of two trains is equal to the ratio of the time they take to pass each other going in the same direction to the time they take to pass each other going in the opposite directions. Find the ratio of the speeds of the two trains.

M6. A city railway network has for sale one-way tickets for travel from one station to another station. Each ticket specifies the origin and destination. Several new stations were added to the network, and an additional 76 different ticket types had to be printed. How many new stations were added to the network?

M7. A circle of radius 6 has an isosceles triangle PQR inscribed in it, where $PQ = PR$. A second circle touches the first circle and the mid-point of the base QR of the triangle as shown. The side PQ has length $4\sqrt{5}$. Find the radius of the smaller circle.



Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

Problem. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation $2x^3 + 9x^2 - 6x - 5 = 0$. What is the third root of the second equation?

(1996 COMC, Problem A1)

Solution 1. The first equation factors as $(x + 5)(x - 1) = 0$. Hence the roots of the first equation are -5 and 1. These are also two roots of the second equation.

Since -5 is a root, we can factor the second equation to get $(x + 5)(2x^2 - x - 1) = 0$. Another root is 1, so we can factor again to make $(x + 5)(x - 1)(2x + 1) = 0$.

Thus, the three roots of the second equation are -5, 1, and $-1/2$.

Hence, the third root of the second equation is $-1/2$.

Solution 2. From the given information, we can deduce that the second equation can be factored into the form $(x^2 + 4x - 5)(ax + b) = 0$ for some numbers a and b.

If we expand this last equation and compare coefficients with the given numbers, we would get $a = 2$ and $-5b = -5$. (This is done by comparing the x^3 terms and the constant terms, respectively.)

The third factor would then be $(2x + 1)$. Hence the third root would be $-1/2$.

Note: For the paranoid, we could check to make sure that $-1/2$ is indeed the third root of the equation. But I will let you do that yourself, because it is late, and I have to work tomorrow :(.

Writer's Guide For Mayhem

Shawn Godin

MATHEMATICAL MAYHEM was conceived as a mathematics journal by students, for students. The original audience of the journal was high school and university students. Since MAYHEM is now contained within CRUX, we, the editors, thought it was time to refine our scope so that we can cut down the overlap between CRUX and MAYHEM. We thought this would be best achieved by having the MAYHEM section focus on a high school audience.

The change in focus to a high school audience does not mean that there will be nothing here for anyone beyond that level; quite the contrary. The only restrictions that we see this change of focus putting on material is the prerequisite knowledge needed by the reader to be able to read it.

Many great mathematics and science writers like Martin Gardner and Ivars Peterson have taken quite complex material and made it accessible to the layperson . It is our hope that items to be published in MAYHEM do not presuppose knowledge beyond the high school level. In cases where some other knowledge is needed, it should be self contained and included in the item.

We seek articles that would be of interest to a wide audience that does not presuppose mathematical background beyond the high school level. The topics do not have to be new, but it is hoped that well known topics will be presented in some new light.

The editors of MAYHEM will be working closely with the editors of CRUX to permit the passing of material from one board to the other as we see an item fitting into the journal. Thus, a piece submitted to CRUX may end up in the MAYHEM section (with the author's approval) or vice versa.

To help to promote MAYHEM, the editors have secured a grant from the CMS endowment fund. Starting in 2002, there will be a wide selection of prizes for solutions to problems and articles submitted and used in MAYHEM. Watch for specific details in future issues.

As we slowly reshape MAYHEM, we want to make sure we are on the right track. Please feel free to contact us at either of the addresses above and let us know how we are doing.

An Extension of Ptolemy's Theorem

David Loeffler

Ptolemy's Theorem is a well known result that states that if $ABCD$ is a convex cyclic quadrilateral (with vertices in this order), then $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

This may be generalized in the following way: let $ABCD$ be any quadrilateral, not necessarily cyclic, not even necessarily convex. Let $A = \angle BAD$ and $C = \angle DCB$.[1] Then we have

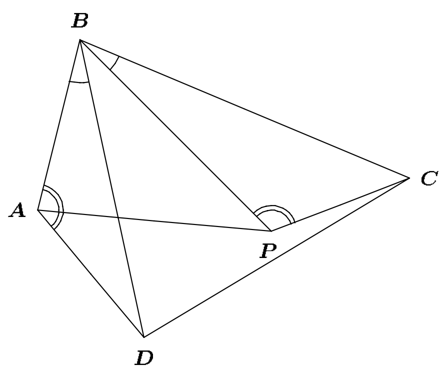
$$AC^2 \cdot BD^2 \;=\; AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A + C)\,.$$

This immediately implies Ptolemy's Theorem, since if $ABCD$ is cyclic, $A + C = \pi$, so that $\cos(A + C) = -1$.

I do not know if this formula is known; I certainly have not seen it before, and I have not met anybody who has.[2]

Proof.

Consider this diagram.



1. If $ABCD$ is not convex, care should be taken that angles are measured in the right sense; that is, so that $\angle BAD$ and $\angle DCB$ have the same orientation.

2. Ed. – See Crux problem 1015 [1985 : 128]: "the following extension of Ptolemy's Theorem has recently appeared (E. Kraemer, Zobecneni Vety Ptolemaivoy, *Rozhledy Mathematickofyzikalni* (Czechoslovakia) 63, no. 8, 345-349):

$$a_{13}^2 a_{24}^2 \;=\; a_{12}^2 a_{34}^2 + a_{23}^2 a_{41}^2 - 2a_{12}a_{23}a_{34}a_{41}\cos(A_1 + A_3)\,."$$

Here the point P is constructed in such a way that $\triangle PBC \sim \triangle ABD$. Thus, we have $\frac{PC}{AD} = \frac{BC}{BD}$, or $BD \cdot PC = AD \cdot BC$.

However, we also have $\angle ABP = \angle CBD$, and by the similarity mentioned above $\frac{AB}{BP} = \frac{BD}{BC}$. Hence, $\triangle ABP$ and $\triangle DBC$ have the same angle at B and the same ratio of the sides adjacent to that angle, so that they are similar. Hence, $\frac{AP}{CD} = \frac{AB}{BD}$, or $BD \cdot AP = AB \cdot CD$.
Furthermore, we have $\angle BPC = \angle BAD = A$, and $\angle APB = \angle DCB = C$, so that $\angle APC = A + C$.

We may now apply the Cosine Rule to $\triangle APC$, obtaining

$$AC^2 \;=\; AP^2 + PC^2 - 2AP \cdot PC \cos(A + C)\,.$$

Multiplying this by BD^2, we have

$$AC^2 \cdot BD^2 \;=\; (BD \cdot AP)^2 + (BD \cdot PC)^2 - 2(BD \cdot AP)(BD \cdot PC)\cos(A + C)\,.$$

Substituting the expressions found above for $BD \cdot AP$ and $BD \cdot PC$, this becomes

$$AC^2 \cdot BD^2 \;=\; AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A + C)\,,$$

as required.

I originally discovered this result while working on a problem from the 1998 IMO shortlist:

Let $ABCDEF$ be a convex hexagon in the plane, with

$$\angle ABC + \angle CDE + \angle EFA \;=\; 2\pi\,.$$

Prove that, if

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA\,,$$

then

$$BC \cdot DF \cdot AE = EF \cdot AC \cdot DB\,.$$

Of course, there are many other solutions to this problem, but one of the simplest purely geometric methods uses the above result.

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Convergent and Divergent Infinite Series

Sandra Pulver

Every irrational number, such as π, e, and $\sqrt{2}$, is the limit of an infinite series. The definite integral, one of the fundamental tools of calculus, is the limit of infinite series.

Given a sequence of numbers

$\{a_n\} : a_1, a_2, a_3, \ldots, a_n, \ldots$

suppose that you form a new sequence $\{s_n\}$ of "partial sums" as follows:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n \\
&\vdots
\end{aligned}
$$

The sequence $\{s_n\}$ is derived from the sequence $\{a_n\}$. If s_n is an infinite series, it is denoted by the symbol

$s_\infty = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Zeno of Elea, a Greek philosopher of the fifth century B.C., demonstrated with a famous series of paradoxes how easily one falls into logical traps when talking about infinite series. How, Zeno asked, can a runner get from A to B? First he must go half the distance. Then he must go half the remaining distance, which brings him to the $\frac{3}{4}$ point. But before completing the last quarter he must again go halfway, to the $\frac{7}{8}$ point. In other words, he goes a distance equal to the sum of the following series:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

How can a runner traverse an infinite series of lengths in a finite period of time? If you keep adding the terms of this series, you will never reach the goal of 1; you are always short by the distance equal to the last fraction added.

The series in the previous problem, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$, the halving series, has a "sum" of 1. In this series, or any infinite series, there is no way to arrive at a "sum" in the usual sense of the word because there is no end to the terms that must be added. The sum — more precisely the limit — of an infinite series, means a number that the value of the series approaches without bound. By "approach" we mean that the difference between the value of the series and its limit can be made as small as one pleases. In every case of an infinite series that has a sum or "converges" one can always find a partial sum that differs from the limit by an amount smaller than any fraction one cares to name. Besides this, in order for an infinite series to have a limit its terms must approach zero.

Consider the halving series again. As the number of terms increases, the last term gets closer to zero. (For example, $\frac{1}{16}$ is closer to zero than $\frac{1}{8}$ is.)

A series is said to be convergent if the sequence of partial sums $\{s_n\}$ has a limit. If $\lim_{n\to\infty} s_n = S$, we say that the series converges to S. If the sequence of partial sums $\{s_n\}$ does not have a limit, the series is said to be divergent. In the case that the series is convergent, the limit S of the series of partial sums $\{s_n\}$ is written

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

This means that the series indicated by the right side converges to the number on the left side.

Consider the following sequence: 10, 50, 250, 1250, $\ldots$. This is called a geometric sequence. This means that every term after the first is obtained by multiplying the term preceding it by a constant, called the common ratio of the sequence. The common ratio of this sequence is 5. The common ratio of the halving series is $\frac{1}{2}$.

Finding the limit of a converging series is often extremely difficult. But when the terms decrease in a geometric progression, as in the halving series, there is a simple method to use. First, let X equal the sum of the entire series. Because each term is twice as large as the next, then multiply each side of the equation by 2:

$$\begin{aligned} 2X &= 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) \\ 2X &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots . \end{aligned}$$

The new series, beyond 1, is the same as the original series X. Thus,

$$2X = 1 + X$$

so that $X = 1$. Therefore, the limit of the halving series is 1.

This method for finding the limit of a converging series where the terms decrease in geometric progression can be applied to another of Zeno's paradoxes: the race of Achilles and the tortoise. Assume that Achilles runs ten times as fast as the tortoise, and that the animal has a lead of 100 yards. After Achilles has gone 100 yards the tortoise has moved 10. After Achilles has run 10 yards the tortoise has moved 1. If Achilles takes the same length of time to run each segment of this series, he will never catch the tortoise, but if they move at uniform speed, he will. How far has Achilles gone by the time he overtakes the tortoise? The answer is the limit of the series

$$100 + 10 + 1 + \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \cdots$$

Each term is ten times the next term. Let X equal the sum of the series, and then multiply each side by 10:

$$10X = 1000 + 100 + 10 + 1 + \tfrac{1}{10} + \tfrac{1}{100} + \cdots$$

This series, after 1000, is the original series. Therefore $10X = 1000 + X$, or $9X = 1000$, and $X = 111\frac{1}{9}$ yards, the number of yards Achilles travels.

This problem can also be solved using another method. Let x equal the distance the tortoise has run. Achilles must run ten times as fast as the tortoise, so Achilles runs a distance of $10x$. You also must consider the tortoise's lead of 100 yards. From this information you get the equation:

$$\begin{aligned} 100 + x &= 10x \\ 100 &= 9x \\ 11\tfrac{1}{9} &= x\,. \end{aligned}$$

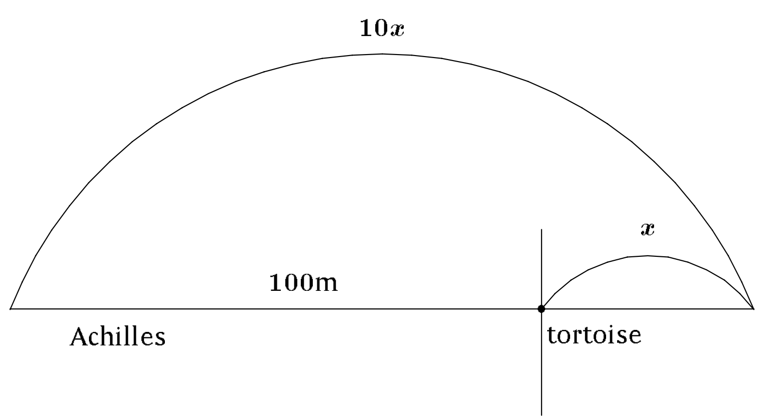

Figure 1.

By solving for x you see that the distance the tortoise has run is $11\frac{1}{9}$ yards. Therefore, since Achilles runs 100 yards more than the tortoise, he overtakes the tortoise after running $111\frac{1}{9}$ yards.

Suppose Achilles runs seven times as fast as the tortoise, which has the same head start of 100 yards. The total distance Achilles travels before overtaking the tortoise is the limit of the series

$$100 + \frac{100}{7} + \frac{100}{7\times 7} + \frac{100}{7\times 7\times 7} + \cdots$$

Each term is seven times the next term. Let x equal the sum of the series; then multiply each side by 7:

$$7x = 700 + 100 + \frac{100}{7} + \frac{100}{7\times 7} + \cdots$$

This series, after 700, is the original series. Therefore $7x = 700 + x$, or $6x = 700$, and $x = 116\frac{2}{3}$, the number of yards Achilles travels.

When no limit, S, exits, a series is called divergent. It is easy to see that $1 + 2 + 3 + 4 + 5 + \cdots$ does not converge. Suppose, however, that each new term in a series joined by a plus sign is smaller than the preceding one. Must such a series converge? It may be hard to believe at first, but the answer is no.

Consider the series known as the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The terms get smaller and smaller; in fact, they approach zero as a limit. Nevertheless, the sum increases without bound. To prove this consider the terms in groups of two, four, eight, and so on, beginning with $\frac{1}{3}$. The first group, $\frac{1}{3} + \frac{1}{4}$, sums to more than $\frac{1}{2}$ because $\frac{1}{3}$ is greater than $\frac{1}{4}$, and a pair of fourths sum to $\frac{1}{2}$. Similarly, the second group, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$, is more than $\frac{1}{2}$ because each term except the last exceeds $\frac{1}{8}$, and a quadruple of eighths sums to $\frac{1}{2}$. In the same way the third group, of eight terms, exceeds $\frac{1}{2}$ because every term except the last $\frac{1}{16}$ is greater than $\frac{1}{16}$, and $\frac{8}{16}$ is $\frac{1}{2}$. Each succeeding group can thus be shown to exceed $\frac{1}{2}$, and since the number of such groups is unlimited the series must diverge.

This can be shown as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)$$

$$+ \left(\frac{1}{17} + \frac{1}{18} + \cdots + \frac{1}{32}\right) + \cdots.$$

Now the harmonic series itself has the sequence of partial sums $\{s_n\}$:

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\vdots \\
s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\
&\vdots
\end{aligned}
$$

Note that the new series, obtained by the insertion of the parentheses, has for its partial sums $s_1, s_2, s_4, s_8, s_{16}, s_{32}, \ldots, s_{2^{n-1}}, \ldots$, of the harmonic series. As said before, the sum in each set of parentheses is larger than $\frac{1}{2}$. By taking enough of these numbers one can make the partial sum of the new series bigger than any positive number, however large. Therefore the series diverges.

The harmonic series diverges very slowly. The first 100 terms, for instance, total only a bit more than 5. To reach 100 requires more than 2^{143} terms, but less than 2^{144} terms. The harmonic series is involved in the following problem. If one brick is placed on another, the greatest offset is obtained by having the centre of gravity of the top brick fall directly above the end of the lower brick, as shown by A in figure 2.

These two bricks, resting on a third, have maximum offset when their combined centre of gravity is above the third brick's edge, as shown by B. By continuing this procedure downward one obtains a column that curves in the manner shown. How large an offset can be obtained? Can it be the full length of a brick?

The answer is that the offset can be as large as one pleases. The top brick projects half a brick's length. The second projects $\frac{1}{4}$, the third $\frac{1}{6}$ and so on down. With an unlimited supply of bricks the offset is the limit of

$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$

This is simply the harmonic series with each term cut in half. Since the sum of the harmonic series can be made larger than any number one cares to name, so can half its sum. In short, the series diverges, and therefore the offset can be increased without limit. Such a series diverges so slowly that it would take a great many bricks to achieve even a small offset.

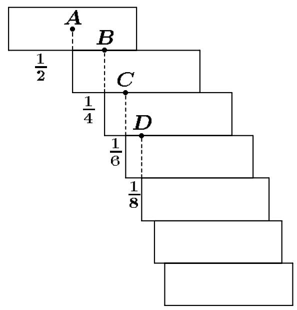

Figure 2.

With 52 playing cards, the first can be placed so that its end is flush with the table edge, and the maximum overhang is a little more than $2\frac{1}{4}$ card lengths.

The harmonic series has many curious properties. If the denominator of each term is raised to the same power n, and n is greater than 1, the series converges. If every other sign, starting with the first, is changed to minus, the resulting series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots .$$

converges to a number slightly smaller than $\frac{7}{10}$.

The following problem introduces the famous series

$$1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots ,$$

which is convergent. The Swiss mathematician Leonhard Euler (1707–1783) discovered in 1763 that the sum of this series is $\frac{\pi^2}{6}$.

If all the terms of an infinite series are positive, it does not matter how the terms are grouped or rearranged; the limit remains the same. But if there are negative terms, it sometimes makes a big difference. From the seventeenth century to the middle of the nineteenth, before laws of limits were carefully formulated, all sorts of disturbing paradoxes were produced by juggling the plus and minus terms of various infinite series. Luigi Guido Grandi, a mathematician at the University of Pisa, considered the simple oscillating series $1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$. If you group the terms $(1-1)+(1-1)+(1-1)+\cdots$, the limit is 0. If you group them $1-(1-1)-(1-1)-(1-1)-\cdots$, and change the signs within the parentheses as required, the sum is 1. This shows, Grandi said, how God could take a universe with parts that added up to nothing and then, by suitable rearrangement, create something. Today, the series is recognized as divergent, so that no meaningful limit or sum can be assigned to it.

Now, consider the series $1 - 2 + 4 - 8 + 16 - \cdots$. Group it $1+(-2+4)+(-8+16)+\cdots$ and you obtain the series $1+2+8+32+\cdots$, which diverges to positive infinity. Group it $(1-2)+(4-8)+(16-32)+\cdots$ and you get the series $-1 - 4 - 16 - 64 - \cdots$ which diverges to infinity in the negative direction.

The climax to all this came in 1854 when Georg Fredrich Bernhard Riemann, the German mathematician now well known for his non-Euclidean geometry, proved a truly remarkable theorem. Whenever the limit of an infinite series can be changed by regrouping or rearranging the order of its terms, it is called conditionally convergent, in contrast to an absolutely convergent series, which is unaffected by such scrambling. Conditionally convergent series always have negative terms, and they always diverge when all their terms have been made positive. Riemann showed that any conditionally convergent series can be suitably rearranged to give a limit that is any desired number whatever, rational or irrational, or even made to diverge to infinity in either direction.

Even an infinite series without negative terms, if it diverges, can cause troubles if you try to handle it with rules that apply only to finite and converging series. For example, let x be the infinite positive sum $1 + 2 + 4 + 8 + 16 + \cdots$. Then $2x$ must equal $2 + 4 + 8 + 16 + \cdots$. This new series is merely the old series minus 1. Therefore $2x = x - 1$, which reduces to $x = -1$. This seems to prove that -1 is infinite and positive, which is not the case. This happens when one uses for divergent series, rules which apply only to infinite converging series with finite sums.

Bibliography

[1.] Bassein, Susan, *An Infinite Series Approach to Calculus*, Houston, TX, Publish or Perish, 1993

[2.] Gardner, Martin, *Martin Gardner's Sixth Book of Mathematical Games from Scientific American*, San Francisco, W.H.Freeman and Company, 1971.

[3.] Gaughan, Edward D., *Sequences and Limits*, Glenview, Scott, Foresman and Company, 1972.

[4.] Moore, Charles, *Summable Series and Convergence Factors*, New York, Dover Publications, 1966.

[5.] O'Brien, Katherine E., *Sequences*, Boston, Houghton Mifflin Company, 1966.

[6.] Rainville, Earl D., *Infinite Series*, New York, Macmillan and Co., 1967.

[7.] Zippin, Leo, *Uses of Infinity*, New York, Random House, Inc., 1962

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, ***please send your proposals and solutions on signed and separate standard*** $8\frac{1}{2}$"×11" ***or*** A4 ***sheets of paper****. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 March 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in* LaTeX *format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

The name of Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina was omitted in error from the list of solvers of problems 2516 and 2522. Our apologies.

2651★. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Professor M.V. Subbarao on the occasion of his 80$^{\text{th}}$ birthday. (Professor Klamkin offers a prize of \$100 for the first correct solution received by the Editor-in-Chief.)

Let P be a non-exterior point of a regular n-dimensional simplex $A_0A_1A_2\ldots A_n$ of edge length e. If

$$F = \sum_{k=0}^{n} PA_k + \min_{0\le k\le n} PA_k\,, \qquad F' = \sum_{k=0}^{n} PA_k + \max_{0\le k\le n} PA_k\,,$$

determine the maximum and minimum values of F and F'.

This problem was suggested by problem 2594 for a general triangle, and the proposer was trying to obtain a stronger inequality by finding the maximum of F.

2652★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let d, e and f be the sides of the triangle determined by the three points at which the internal angle-bisectors of given $\triangle ABC$ meet the opposite sides. Prove that

$$d^2 + e^2 + f^2 \;\leq\; \frac{s^2}{3}\,,$$

where s is the semiperimeter of $\triangle ABC$.

Show also that equality occurs if and only if the triangle is equilateral.

2653. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For whole numbers $n \geq 0$ and $N \geq 1$, evaluate the (combinatorial) sum

$$S_N(n) := \sum_{k \geq n} \binom{N}{2k}\binom{k}{n}\,.$$

2654 *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $\triangle ABC$ has medians AD, BE and CF. Suppose that L, M and N are points on the sides BC, CA and AB respectively.

Prove that the line through L parallel to AD, the line through M parallel to BE and the line through N parallel to CF are concurrent if and only if

$$\frac{BL^2}{BC^2} + \frac{CM^2}{CA^2} + \frac{AN^2}{AB^2} \;=\; \frac{LC^2}{BC^2} + \frac{MA^2}{CA^2} + \frac{NB^2}{AB^2}\,.$$

2655 *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b and c be the sides of $\triangle ABC$ and let s be its semiperimeter. Given that

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \;=\; s\,,$$

show that $\triangle ABC$ is equilateral.

2656★. *Proposed by Vedula N. Murty, Dover, PA, USA.*

For positive real numbers a, b and c, show that

$$\frac{(1-b)(1-bc)}{b(1+a)} + \frac{(1-c)(1-ca)}{c(1+b)} + \frac{(1-a)(1-ab)}{a(1+c)} \;\geq\; 0\,.$$

2657. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Prove that

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right) \;=\; \frac{2k}{1 + (-1)^{k+1}\sqrt{2}\sin\theta}\,.$$

2658. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ have $\angle BCA = 90^\circ$. Squares $ACDE$ and $CBGF$ are drawn externally to the triangle. Suppose that AG and BE intersect at M. Show that M lies on the altitude CN.

2659. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, the side BC is fixed. A is a variable point. Assume that $AC > AB$. Let M be the mid-point of BC, let O be the circumcentre of $\triangle ABC$, let R be the circumradius, let G be the centroid and H the orthocentre. Assume that the Euler line, OH, is perpendicular to AM.

1. Determine the locus of A.
2. Determine the range of $\angle BGC$.

2560. . *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let z_1, z_2, ..., z_n be distinct non-zero complex numbers. Prove that

$$\sum_{j=1}^{n} z_j^{n-1}\left(1+\prod_{\substack{k=1\\k\neq j}}^{n} z_k\right)\prod_{\substack{k=1\\k\neq j}}^{n}\frac{1}{z_k-z_j}$$

is a real number, and determine its value.

2561. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let H be the orthocentre of acute-angled $\triangle ABC$ in which $\tan\left(\frac{A}{2}\right) = \frac{1}{2}$. Show that the sum of the radii of the incircles of $\triangle AHB$ and $\triangle AHC$ is equal to the inradius of $\triangle ABC$.

Is the converse true?

2562. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $\triangle ABC$ is acute-angled, has inradius r and has area Δ. Prove that

$$\left(\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\right)^2 \leq \frac{\Delta}{r^2}.$$

2563. *Proposed by Antreas P. Hatzipolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that the incircle of $\triangle ABC$ is tangent to the circle with BC as diameter. Show that the excircle on BC has radius equal to BC.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2542. [2000 : 237] *Proposed by Hassan A. Shah Ali, Tehran, Iran.*

Suppose that k is a natural number and $\alpha_i \geq 0$, $i = 1, \ldots, n$, and $\alpha_{n+1} = \alpha_1$. Prove that

$$\sum_{\substack{1\leq i\leq n\\ 1\leq j\leq k}} \alpha_i^{k-j}\alpha_{i+1}^{j-1} \;\geq\; \frac{k}{n^{k-2}}\left(\sum_{1\leq i\leq n} \alpha_i\right)^{k-1} .$$

Determine the necessary and sufficient conditions for equality.

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and Henry Liu, Trinity College, Cambridge, England.

By the Power-Mean Inequality, if $a_i \geq 0$ for $i = 1, \ldots, m$, and $r \in \mathbb{N} \cup \{0\}$, then

$$\sum_{i=1}^{m} a_i^r \;\geq\; \frac{1}{m^{r-1}}\left(\sum_{i=1}^{m} a_i\right)^r . \tag{1}$$

We have equality if and only if $r = 0$, or $r = 1$, or $r \geq 2$ and $a_1 = \cdots = a_m$.

Next, we show that the given inequality holds for $n = 2$; that is, if $b_1, b_2 \geq 0$, then

$$\sum_{j=1}^{k} b_1^{k-j}b_2^{j-1} + \sum_{j=1}^{k} b_2^{k-j}b_1^{j-1} = 2\sum_{j=1}^{k} b_1^{k-j}b_2^{j-1} \geq \frac{k}{2^{k-2}}(b_1+b_2)^{k-1}. \tag{2}$$

We will use induction on k. If $k = 1$, then both sides of the inequality (2) are equal to 2 (or if $k = 2$, then both sides are equal to $2(b_1 + b_2)$). Suppose the inequality (2) holds for some $k \geq 2$. Multiplying (2) by b_1, then by b_2, and adding the two inequalities, we obtain

$$2\sum_{j=1}^{k} b_1^{k+1-j}b_2^{j-1} + 2\sum_{j=1}^{k} b_1^{k-j}b_2^{j} \;\geq\; \frac{k}{2^{k-2}}(b_1+b_2)^{k-1}b_1 + \frac{k}{2^{k-2}}(b_1+b_2)^{k-1}b_2$$

or

$$b_1^k + \sum_{j=2}^{k} b_1^{k+1-j}b_2^{j-1} + \sum_{j=1}^{k-1} b_1^{k-j}b_2^{j} + b_2^k \;\geq\; \frac{k}{2^{k-1}}(b_1+b_2)^k .$$

By (1), $b_1^k + b_2^k \geq \frac{1}{2^{k-1}}(b_1+b_2)^k$. Thus,

$$b_1^k + \sum_{j=2}^{k} b_1^{k+1-j} b_2^{j-1} + \sum_{j=1}^{k} b_1^{k-j} b_2^j + b_2^k + b_1^k + b_2^k \geq \frac{k+1}{2^{k-1}}(b_1+b_2)^k$$

or

$$2\sum_{j=1}^{k+1} b_1^{k+1-j} b_2^{j-1} \geq \frac{k+1}{2^{k-1}}(b_1+b_2)^k .$$

The induction is complete, so that the inequality (2) is true. Equality in (2) holds if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $b_1 = b_2$. Applying (1) and (2), we have

$$\begin{aligned}
\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \alpha_i^{k-j}\alpha_{i+1}^{j-1} &= \frac{1}{2}\sum_{i=1}^{n}\left(2\sum_{j=1}^{k}\alpha_i^{k-j}\alpha_{i+1}^{j-1}\right) \\
&\geq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{k}{2^{k-2}}(\alpha_i+\alpha_{i+1})^{k-1}\right) \qquad (3)\\
&\geq \frac{k}{2^{k-1}}\left(\frac{1}{n^{k-2}}\left(\sum_{i=1}^{n}(\alpha_i+\alpha_{i+1})\right)^{k-1}\right) \qquad (4)\\
&= \frac{k}{2^{k-1}}\left(\frac{2^{k-1}}{n^{k-2}}\left(\sum_{i=1}^{n}\alpha_i\right)^{k-1}\right) \\
&= \frac{k}{n^{k-2}}\left(\sum_{i=1}^{n}\alpha_i\right)^{k-1} ,
\end{aligned}$$

as required.

The cases of equality are determined from (3): it holds if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $\alpha_1 = \cdots = \alpha_n$. (In (4), we have equality if and only if $k = 1$, or $k = 2$, or $k \geq 3$ and $\alpha_1 + \alpha_2 = \cdots = \alpha_n + \alpha_1$, which is a weaker condition.)

Also solved by AUSTRIAN IMO-TEAM 2000; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

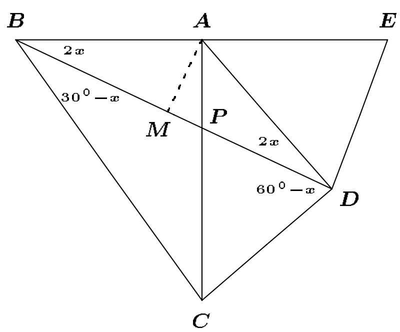
2543. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

In quadrilateral $ABCD$, we have $\angle ABD = \angle ADB = \angle BDC = 40°$, $\angle DBC = 10°$ and AC and BD meet at P. Show that $BP = AP + PD$.

Editor's comment. The majority of solvers used the Sine Law (sometimes with the Cosine Law) to prove the result. A smaller number of solutions relied only on elemetary geometry to prove the result. We present the solution by T. Seimiya, who used elementary means to prove a more general result.

Solution by Toshio Seimiya, Kawasaki, Japan.

In quadrilateral $ABCD$, we have $\angle ABD = \angle ADB = 2x$, $\angle BDC = 60^\circ - x$, $\angle DBC = 30^\circ - x$ (where $0^\circ < x < 30^\circ$), and AC and BD meet at P. Show that $BP = AP + PD$. (When $x = 20^\circ$, we obtain problem 2543.)



Since $\angle ABD = \angle ADB$, we have that $AB = AD$. Let E be the point on BA produced beyond A such that $AE = AB = AD$. Then

$$\begin{aligned}\angle BED &= \tfrac{1}{2}\angle BAD \\ &= \tfrac{1}{2}(180^\circ - 2x - 2x) \\ &= 90^\circ - 2x\,.\end{aligned}$$

Since

$$\angle BCD = 180^\circ - (\angle CBD + \angle CDB) = 180^\circ - (90^\circ - 2x) = 90^\circ + 2x\,,$$

we have $\quad \angle BED + \angle BCD = (90^\circ - 2x) + (90^\circ + 2x) = 180^\circ\,.$

Hence B, C, D and E lie on a circle.

Since $AB = AD = AE$, we see that A is the centre of this circle, and so $\angle CAD = 2\angle CBD = 2\,(30^\circ - x) = 60^\circ - 2x$.

Thus, $\angle APB = \angle PAD + \angle ADB = (60^\circ - 2x) + 2x = 60^\circ$.

Now let M be the foot of the perpendicular from A to BD. Since $AB = AD$, we see that M is the mid-point of BD.

Hence, $\quad BP - PD = 2MP\,. \quad (1)$

Since $APM = 60^\circ$, we get $\quad AP = 2PM\,. \quad (2)$

From (1) and (2), we deduce that $AP = BP - PD$, and therefore, that $BP = AP + PD$.

Also solved by HAYO AHLBURG, Benidorm, Spain; AUSTRIAN IMO TEAM 2000; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; GERRY

LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2544. [2000 : 237] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

For any triangle ABC, find the exact value of

$$\sum_{\text{cyclic}} \frac{\cos A + \cos B}{1 + \cos A + \cos B - \cos C}\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

From

$$\begin{aligned}\frac{\cos A + \cos B}{\sin A + \sin B} &= \frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} \\ &= \cot\left(\tfrac{A+B}{2}\right) = \tan\left(\tfrac{C}{2}\right)\end{aligned}$$

we get $\cos A + \cos B = (\sin A + \sin B)\cdot\tan\left(\frac{C}{2}\right)$.

Since $1 - \cos C = 2\sin^2\left(\frac{C}{2}\right) = \sin C\cdot\tan\left(\frac{C}{2}\right)$ and $\tan\left(\frac{C}{2}\right) \neq 0$, the given sum becomes

$$\sum_{\text{cyclic}} \frac{\sin A + \sin B}{\sin A + \sin B + \sin C} = 2\,.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RAJPARTAP KHANGURA, student, Angelo State University, San Angelo, TX, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; R. LAUMEN, Deurne-Antwerp, Belgium; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Visakhapatnam, India; HENRY PAN , Student, East York C.I. Toronto; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

About one third of the submitted solutions used the fact, with or without proof, that $\sum_{\text{cyclic}} \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = 1$ *when* $A + B + C = \pi$.

2545. [2000 : 237] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

For any triangle ABC, prove that

$$\sum_{\text{cyclic}} \frac{\sin^3 A}{\sin A\,(-\cos^2 A + \cos^2 B + \cos^2 C) + \sin B \cos(A-B) + \sin C \cos(A-C)} \;=\; 1\,.$$

Solution by Richard B. Eden, Ateneo de Manila University, Philippines.

We are going to use the identities $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta$ and $\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cos\beta$.

First,

$$\begin{aligned}
&\sin B\cos(A-B) + \sin C\cos(A-C) \\
&\quad= \tfrac{1}{2}\left[\sin A + \sin(2B-A) + \sin A + \sin(2C-A)\right] \\
&\quad= \sin A + \left[\sin(B+C-A)\cos(B-C)\right] \\
&\quad= \sin A + \sin(180-2A)\cos(B-C) \\
&\quad= \sin A + \sin 2A\cos(B-C) = \sin A\left[1 + 2\cos A\cos(B-C)\right] \\
&\quad= \sin A\left[1 - 2\cos(B+C)\cos(B-C)\right] \\
&\quad= \sin A\left[1 - \cos 2B - \cos 2C\right] \\
&\quad= \sin A\left[3 - 2\cos^2 B - 2\cos^2 C\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\text{cyclic}} \frac{\sin^3 A}{\sin A(-\cos^2 A + \cos^2 B + \cos^2 C) + \sin B\cos(A-B) + \sin C\cos(A-C)} \\
&\quad= \sum_{\text{cyclic}} \frac{\sin^2 A}{(-\cos^2 A + \cos^2 B + \cos^2 C) + (3 - 2\cos^2 B - 2\cos^2 C)} \\
&\quad= \sum_{\text{cyclic}} \frac{\sin^2 A}{3 - \cos^2 A - \cos^2 B - \cos^2 C} \;=\; \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 A + \sin^2 B + \sin^2 C} \;=\; 1\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RAJPARTAP KHANGURA, student, Angelo State University, San Angelo, TX, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ANDY LIU, University of Alberta, Edmonton, Alberta; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU, Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most solutions were variations of the above one. Three solvers started with the Law of Sines. Four others gave abbreviated solutions, making use of the indentity

$$\sum_{\text{cyclic}} \sin^2 A \;=\; 2(1 + \cos A\cos B\cos C)\,.$$

2546. [2000 : 237] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Prove that triangle ABC is equilateral if and only if

$$a\cos(B-C)+b\cos(C-A)+c\cos(A-B) \;=\; \frac{a^4+b^4+c^4}{abc}\,.$$

Solution by Henry Liu, student, Trinity College, Cambridge, England.

Obviously, if $\triangle ABC$ is equilateral, then both sides of the given expression are equal to $3a$.

Conversely, suppose that the given equality holds. By repeated use of the AM-GM inequality,

$$\begin{aligned} a^4+b^4+c^4 &= \frac{a^4+b^4}{2}+\frac{b^4+c^4}{2}+\frac{c^4+a^4}{2} \\ &\geq a^2b^2+b^2c^2+c^2a^2 \\ &= a^2\left(\frac{b^2+c^2}{2}\right)+b^2\left(\frac{c^2+a^2}{2}\right)+c^2\left(\frac{a^2+b^2}{2}\right) \\ &\geq a^2bc+b^2ca+c^2ab \\ &= abc(a+b+c)\,. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{a^4+b^4+c^4}{abc} &\geq a+b+c \\ &\geq a\cos(B-C)+b\cos(C-A)+c\cos(A-B) \\ &= \frac{a^4+b^4+c^4}{abc}\,. \end{aligned}$$

Thus, we must have equality throughout. It is easy to see that

$$\frac{a^4+b^4+c^4}{abc} \;=\; a+b+c$$

if and only if $a=b=c$, and

$$a+b+c \;=\; a\cos(B-C)+b\cos(C-A)+c\cos(A-B)$$

if and only if $\cos(B-C)=\cos(C-A)=\cos(A-B)=1$; that is, if and only if $A=B=C=\frac{\pi}{3}$. Therefore, $\triangle ABC$ is equilateral, which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ALEXANDER CORNELIUS, student, Angelo State University, San Angelo, TX, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Manila, the

Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; R. LAUMEN, Belgium; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most of the solvers showed that the given equality is equivalent to either

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 \;=\; 0$$

or some trigonometric equality, from which the result follows immediately. Lee seems to have found the fanciest equivalent form:

$$a\,(1-\cos(B-C))+b\,(1-\cos(C-A))+c\,(1-\cos(A-B)) + \frac{1}{2abc}\sum_{\text{cyclic}}(a-b)^2[(a+b)^2+c^2] \;=\; 0\,.$$

2547. [2000 : 238] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In triangle ABC with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$, and area Δ, prove that

$$\frac{a^2+b^2+c^2}{\Delta} \;=\; 4\sqrt{7}\,.$$

Solution by Richard Eden, Ateneo de Manila University, Manila, the Philippines.

Let $\angle A=\frac{\pi}{7}$, $\angle B=\frac{2\pi}{7}$ and $\angle C=\frac{4\pi}{7}$. By the Law of Cosines,

$$a^2 \;=\; b^2+c^2-2bc\cos\frac{\pi}{7}\,.$$

Also, $\Delta=\frac{1}{2}bc\sin\frac{\pi}{7}$, so that

$$a^2 \;=\; b^2+c^2-2\left(\frac{2\Delta}{\sin\frac{\pi}{7}}\right)\cos\frac{\pi}{7} \;=\; b^2+c^2-4\Delta\cot\frac{\pi}{7}\,,$$

Similarly, $$b^2 \;=\; c^2+a^2-4\Delta\cot\frac{2\pi}{7}$$

and $$c^2 \;=\; a^2+b^2-4\Delta\cot\frac{4\pi}{7}\,.$$

Adding these three equalities and dividing by Δ yields,

$$\frac{a^2+b^2+c^2}{\Delta} \;=\; 4\left(\cot\frac{\pi}{7}+\cot\frac{2\pi}{7}+\cot\frac{4\pi}{7}\right)\,.$$

By problem 2537, $\cot\frac{\pi}{7}+\cot\frac{2\pi}{7}-\cot\frac{3\pi}{7}=\sqrt{7}$.

Since $\cot\frac{4\pi}{7}=-\cot(\pi-\frac{4\pi}{7})$, we obtain

$$\frac{a^2+b^2+c^2}{\Delta} = 4\sqrt{7}\,.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian College, South Carolina, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA (two solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece (two solutions); D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2548*. [2000 : 238] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana, USA.*

Let $a(1)=1$ and, for $n\geq 2$, define $a(n)=\lfloor a(n-1)/2\rfloor$, if this is not in $\{0,\,a(1),\,\ldots,\,a(n-1)\}$, and $a(n)=3a(n-1)$ otherwise.

(a) Does any positive integer occur more than once in this sequence?

(b) Does every positive integer occur in this sequence?

Editorial comment.

There were no solutions received to either part. One reader (Richard Hess) reports that he has checked the first 15000 terms of the sequence by computer, and that there are no repeated terms and all positive integers up to 1435 appear. The problem remains open, and perhaps hopeless, though a solution to the first part may be possible.

The proposer also invites the readers to try replacing the divisor 2 and the multiplier 3 by any pair of relatively prime integers greater than 1, to see if the same behaviour seems to occur.

2549*. [2000 : 238] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Is it possible to choose four points in the plane such that all the distances that they determine are odd integers?

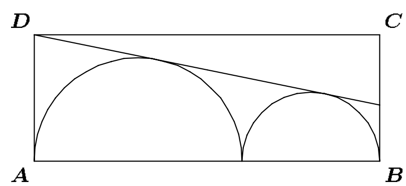
Editorial comment.

This problem appeared as problem B5 of the 1993 Putnam Competition. Thus a solution of it has been published in the October 1994 *American Mathematical Monthly*, pages 733–734. We thank John Leonard of the University of Arizona for this information. The proposer has noted that the problem follows from an even earlier *Monthly* article by Graham, Rothschild and Straus ("Are there $n+2$ points in E^n with odd integral distances?", Vol. **81**(1974), pp. 21–25). Incidentally, according to this paper, the answer to the question in the title is "yes" if and only if $n \equiv -2 \pmod{16}$!

Solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JASON YOUNG, student, University of Arizona, Tucson, Arizona, USA. Young's solution was found without knowledge of the problem's earlier appearance in the Putnam.

2550. [2000 : 238] *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

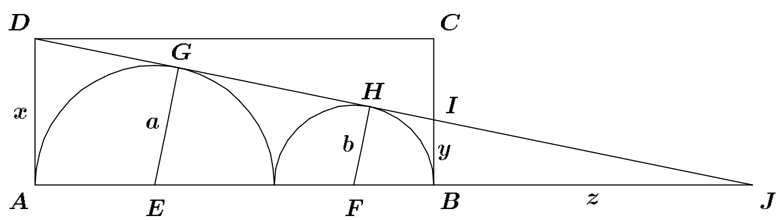
Given are two semi-circles, C_a and C_b of different radii a and b, and a rectangle $ABCD$ such that the diameters of the semi-circles lie contiguously on the side AB as shown, and the common tangent to the semi-circles passes through the vertex D of the rectangle.



Find, in terms of a and b, the ratio in which the common tangent divides the side BC.

Solution by Jesse Crawford, student, Angelo State University, San Angelo, TX, USA.

Let C_a and C_b be the semi-circles containing the points A and B, respectively, let E and F be the centres of C_a and C_b, respectively, and let G and H be the points of tangency of C_a and C_b with the common tangent line, respectively. Let I be the intersection of the common tangent with the line CB, and let J be the intersection of the extensions of DI and AB.



Since DJ is tangent to the semi-circles, we have $\angle EGJ = \angle FHJ = 90^\circ$. It follows that $\triangle ADJ \sim \triangle BIJ$ and $\triangle GEJ \sim \triangle HFJ$. Let x, y and z be the lengths of DA, IB and BJ, respectively. Using proportionality of the sides of similar triangles, we have

$$\frac{y}{x} = \frac{z}{2a+2b+z}$$

and

$$\frac{a}{b} = \frac{a+2b+z}{b+z}.$$

Eliminating z yields $$\frac{y}{x} = \frac{b^2}{a^2}.$$ ■

Also solved by HAYO AHLBURG, Benidorm, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, Quebec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student, Cummer Valley Middle School, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; DANIEL REISZ, IREM, Université de Bourgogne, Vincelles, France; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; DARKO VELJAN, University of Zagreb, Zagreb, Croatia; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most solvers showed the equivalent result that the ratio of the two parts of CD was $\dfrac{a^2-b^2}{b^2}$.

Diminnie noted that if $\frac{a}{b} = \phi$, the Golden Ratio $\frac{1+\sqrt{5}}{2}$, then $\frac{CI}{IB}$ is also the Golden Ratio.

2551. [2000 : 303] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Suppose that a_k $(1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

Prove that

$$\prod_{j=1}^{n} \sum_{k=1}^{n} e_{j,k} a_k^2 \ \ge\ \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2 .$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

On expanding out, the inequality reduces to

$$\prod_{k=1}^{n} \left[(n-2)S + a_k^2\right] \;\geq\; \prod_{k=1}^{n} (T - a_k)^2$$

where $S = \sum_{k=1}^{n} a_k^2$ and $T = \sum_{k=1}^{n} a_k$.

Since $(T - a_1)^2 \leq (n-1)(S - a_1^2)$, etc., it suffices to prove that

$$\prod_{k=1}^{n} \left[(n-2)S + a_k^2\right] \;\geq\; (n-1)^n \prod_{k=1}^{n} \left(S - a_k^2\right) . \qquad (1)$$

If we now let $x_k = S - a_k^2$, where $k = 1, 2, \ldots, n$, so that

$$S = (x_1 + x_2 + \cdots + x_n)/(n-1) \text{ and } a_k^2 = S - x_k ,$$

(1) becomes

$$\prod_{k=1}^{n} (S' - x_k) \;\geq\; (n-1)^n x_1 x_2 \cdots x_n , \text{ where } S' = x_1 + x_2 + \cdots x_n .$$

The result now follows by applying the AM–GM inequality to each of the factors $(S' - x_k)$ on the left-hand side. There is equality if and only if all the a_k's are equal.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, Trinity College, Cambridge, England; and the proposer.

Janous notes that by employing the general Power-Mean Inequality, we can prove more: that for $\alpha \geq 1$

$$\prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^{\alpha} \;\leq\; (n-1)^{n(\alpha-2)} \prod_{j=1}^{n} \left(\sum_{j=1}^{n} e_{j,k} a_k^{\alpha} \right) .$$

2552. [2000 : 303] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Suppose that $a, b, c > 0$. If $x \geq \frac{a+b+c}{3\sqrt{3}} - 1$, prove that

$$\frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} \;\geq\; abc .$$

Solution by Michel Bataille, Rouen, France.

The condition on x can be rewritten as

$$(x+1)^2 \geq \frac{(a+b+c)^2}{27}. \tag{1}$$

The AM–GM inequality gives

$$\frac{(a+b+c)^3}{27} \geq abc. \tag{2}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\left(\frac{b+cx}{\sqrt{a}}\right)^2 + \left(\frac{c+ax}{\sqrt{b}}\right)^2 + \left(\frac{a+bx}{\sqrt{c}}\right)^2\right)\left((\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2\right) \\ &\geq \ ((b+cx)+(c+ax)+(a+bx))^2 \\ &= \ (a+b+c)^2(x+1)^2 . \end{aligned}$$

Hence

$$\frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} \geq (a+b+c)(x+1)^2 .$$

Using (1) and (2), we obtain

$$\begin{aligned} \frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} &\geq (a+b+c)\frac{(a+b+c)^2}{27} \\ &\geq abc , \end{aligned}$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; HENRI LIU, Trinity College, Cambridge, England; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFLER, student, Gotham School, Bristol, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; IVAN SLAVOV, student, English Language High School, Stara Zagora, Bulgaria; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Most of the proposed solutions are similar to the one given above. Bencze, Seiffert and Slavov have suggested various generalizations of the given inequality.

2553. [2000 : 303] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find all real roots of the equation

$$\frac{\left(\sqrt{2x^2-2x+12} - \sqrt{x^2-5}\right)^3}{(5x^2-2x-3)\sqrt{2x^2-2x+12}} = \frac{2}{9} .$$

I. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let x be a real root of the equation. Then

$$\frac{(u-v)^3}{(u^2+3v^2)u} \;=\; \frac{2}{9}\,, \qquad (1)$$

where $u = \sqrt{2x^2-2x+12}$ and $v = \sqrt{x^2-5}$. Obviously, $v \neq 0$ so that (1) may be written as

$$\frac{(w-1)^3}{(w^2+3)w} \;=\; \frac{2}{9}\,,$$

where $w = u/v$. After clearing the fractions and expanding, this becomes

$$7w^3 - 27w^2 + 21w - 9 \;=\; 0\,,$$

or, equivalently,

$$(w-3)(7w^2-6w+3) \;=\; 0\,.$$

The equation $7w^2 - 6w + 3 = 0$ has no real roots. Hence we must have $w = 3$. From $u = 3v$, we have $u^2 = 9v^2$ or $7x^2 + 2x - 57 = 0$. Thus, $x = -3$ and $x = 19/7$. It is easily verified that both of these are solutions, which means they are the only solutions.

II. *Solution by Michel Bataille, Rouen, France.*

Suppose that x is such a root and let $a = \sqrt{2x^2-2x+12}$ and $b = \sqrt{x^2-5}$. Observing that $5x^2 - 2x - 3 = a^2 + 3b^2$, we get

$$\frac{2}{9} \;=\; \frac{(a-b)^3}{a(a^2+3b^2)} \;=\; \frac{2(a-b)^3}{(a+b)^3+(a-b)^3}\,.$$

Hence $a \neq b$ and

$$1 + \left(\frac{a+b}{a-b}\right)^3 \;=\; 9\,,$$

from which $a = 3b$ is easily obtained. Since $a^2 = 9b^2$ may be written as $7x^2 + 2x - 57 = 0$, we must have $x = -3$ or $x = 19/7$. Conversely, when $x = -3$ or $x = 19/7$, it is easy to check that $2x^2 - 2x + 12$ and $x^2 - 5$ are positive or that $a = 3b$. This shows that these values are indeed solutions. In conclusion, there are exactly two real roots, namely -3 and $19/7$.

Also solved by ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD EDEN, Ateneo de Manila University, Manilla, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; IVAN SLAVOV, Stara Zagora, Bulgaria; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

2554. [2000 : 304] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In triangle ABC, prove that at least one of the quantities

$$(a+b-c)\tan^2\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right),$$
$$(-a+b+c)\tan^2\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right),$$
$$(a-b+c)\tan^2\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right),$$

is greater than or equal to $\frac{2r}{3}$, where r is the radius of the incircle of $\triangle ABC$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, (adapted by the editor).

We shall prove a stronger result; that is, the arithmetic mean, M, of the three given quantities is at least $2r/3$. Let $s = \frac{1}{2}(a+b+c)$ denote the semiperimeter of the triangle and let $x = s-a$, $y = s-b$, $z = s-c$. Then clearly x, y, and z are all positive with $x+y+z = s$.

By well-known formulas we have:

$$\begin{aligned} r &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{xyz}{x+y+z}}, \\ \tan\left(\frac{A}{2}\right) &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \text{etc.} \end{aligned}$$

Thus,

$$\begin{aligned} &(-a+b+c)\ \tan^2\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) \\ &\qquad = 2(s-a)\frac{(s-c)(s-a)}{s(s-b)}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &\qquad = \frac{2x^2z}{(x+y+z)y}\sqrt{\frac{xy}{(x+y+z)z}} = \frac{2x^2r}{y(x+y+z)}. \end{aligned}$$

Similarly,

$$(a-b+c)\tan^2\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) = \frac{2y^2r}{z(x+y+z)}$$

and

$$(a+b-c)\tan^2\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = \frac{2z^2r}{x(x+y+z)}.$$

Hence, $M \geq \frac{2r}{3}$ is equivalent to

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z$$

which follows from the Cauchy-Schwarz Inequality since

$$\begin{aligned}\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right)(y + z + x) &\geq \left(\frac{x}{\sqrt{y}}\sqrt{y} + \frac{y}{\sqrt{z}}\sqrt{z} + \frac{z}{\sqrt{x}}\sqrt{x}\right)^2 \\ &= (x + y + z)^2 \ .\end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; RICHARD B. EDEN, Ateneo de Manila University, Philippines; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFER, student, Cotham School, Bristol, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; IVAN SLAVOV, student, Stara Zagora, Bulgaria; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were one incorrect and one incomplete solutions.

Besides Janous, Woo also proved the stronger result by very similar argument.