SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The name of O. Ciaurri, Universidad de La Roija, Logroño, Spain, a co-solver with Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain, was unfortunately omitted in the solution to problem 2404. Our apologies.

2436. [1999: 173, 191] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] \;=\; 0\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and by Heinz-Jürgen Seiffert, Berlin, Germany (combined by the editor).

Correction: instead of

$$t^p + t^{-p} + 2^p \;=\; t + t^{-1} + 2\,.$$

read

$$t^p + t^{-p} + 2^p \;=\; \left(t + t^{-1}\right)^p + 2\,.$$

2451. [1999: 306] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Construct an infinite sequence, $\{A_n\}$, of infinite subsets of $\mathbb{N}$ with the following properties:

(a) the intersection of any two distinct sets A_n and A_m is a singleton;

(b) the singleton in (a) is a different one if at least one of the distinct sets A_n, A_m, is changed (so the new pair is again distinct);

(c) every natural number is the intersection of (exactly) one pair of distinct sets as in (a).

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

Arrange all the natural numbers in an infinite triangular array $\{a_{ij}\}$, $1 \le i \le j$, filling column n with n consecutive integers, as follows:

$$\begin{array}{cccccccc}
1 & 2 & 4 & 7 & 11 & 16 & \dots \\
 & 3 & 5 & 8 & 12 & 17 & \dots \\
 & & 6 & 9 & 13 & 18 & \dots \\
 & & & 10 & 14 & 19 & \dots \\
 & & & & 15 & 20 & \dots \\
 & & & & & 21 & \dots \\
 & & & & & & \ddots
\end{array}$$

Then define:

$$\begin{array}{lcll}
A_0 & = & \{1, 3, 6, 10, 15, \dots\} & \text{(main diagonal)} \\
A_1 & = & \{1, 2, 4, 7, 11, 16, \dots\} & \text{(first row)} \\
A_2 & = & \{2, 3, 5, 8, 12, 17, \dots\} & \text{(union of column 2 and row 2)} \\
 & & \dots & \\
A_n & = & \text{union of column } n \text{ and row } n & \\
 & & \dots &
\end{array}$$

Since $A_0 \cap A_i = \{a_{ii}\}$ for all $1 \le i$ and $A_i \cap A_j = \{a_{ij}\}$ for all $1 \le i < j$, it is clear that (a), (b) and (c) hold.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER and DOUG CASHING, St. Bonaventure University, St. Bonaventure, NY, USA; DEE SNELL, University of Wisconsin, La Crosse, WI, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Dergiades, Sealy, Sedinger and Cashing, Snell, and Young all gave solutions similar to the above.

2452. [1999: 307, 428] *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!}.$$

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^4}{(n+1)!}.$$

(c) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^6}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^7}{(n+1)!}.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA. (modified slightly by the editor).

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2n+1}{(2n)!} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n)!}$

$= \sinh 1 + \cosh 1 = e$, while

$$\sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!} = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

$= \cosh 1 + \sinh 1 = e.$

(b) and (c). For $k \geq 1$, let

$$I_k = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^k}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^{k-1}}{(n)!}.$$

It can be shown easily by the Ratio Test that I_k converges absolutely for all $k \geq 1$.

Note that $I_1 = \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} = \frac{1}{e} - 1$, while for $k \geq 2$,

$$I_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{i=0}^{k-1} \binom{k-1}{i} n^i = \sum_{i=0}^{k-1} \binom{k-1}{i} \sum_{n=1}^{\infty} \frac{(-1)^n n^i}{n!}$$

$$= \sum_{i=0}^{k-1} \binom{k-1}{i} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)^i}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} - \sum_{i=1}^{k-1} \binom{k-1}{i} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^i}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} - \sum_{i=1}^{k-1} \binom{k-1}{i} (1 + I_i)$$

$$= \left(\frac{1}{e} - 1\right) - \sum_{i=1}^{k-1} \binom{k-1}{i} - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i$$

$$= \left(\frac{1}{e} - 1\right) - (2^{k-1} - 1) - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i$$

$$= \frac{1}{e} - 2^{k-1} - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i .$$

Define $\{a_k\}$ by $a_1 = 1$ and $a_k = 1 - \sum_{i=1}^{k-1} \binom{k-1}{i} a_i$ for $k \geq 2$. We show by induction that $I_k = \frac{a_k}{e} - 1$ for all $k \geq 1$.

Since $I_1 = \frac{1}{e} - 1$, the claim is true for $k = 1$.

Suppose $I_k = \frac{a_k}{e} - 1$ for some $k \geq 1$. Then using the recurrence relation obtained above, we have

$$I_{k+1} = \frac{1}{e} - 2^k - \sum_{i=1}^{k} \binom{k}{i} I_i = \frac{1}{e} - 2^k - \sum_{i=1}^{k} \binom{k}{i} \left(\frac{a_i}{e} - 1\right)$$

$$= \frac{1}{e} - 2^k - \frac{1}{e} \sum_{i=1}^{k} \binom{k}{i} a_i + \sum_{i=1}^{k} \binom{k}{i}$$

$= \frac{1}{e} - 2^k - \frac{1}{e} \left(1 - a_{k+1}\right) + \left(2^k - 1\right) = \frac{a_{k+1}}{e} - 1$, completing the induction.

The first ten terms of the sequence $\{a_k\}$ are:

$$1,\ 0,\ -1,\ -1,\ 2,\ 9,\ 9,\ -50,\ -167,\ -513,\ \cdots .$$

Hence it follows immediately that $I_3 = \frac{-1}{e} - 1 = I_4$ and $I_6 = \frac{9}{e} - 1 = I_7$.

Also solved by MICHEL BATAILLE, Rouen, France; THE BOOKERY PROBLEM GROUP, Walla Walla, WA, USA ; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); and the proposer.

Actually, as pointed out by Janous, all these results were contained in the proposer's paper "Apropos Bell and Stirling Numbers" *which appeared in* **CRUX with MAYHEM** [1999 : 274-281].

Diminnie commented that it would be interesting to know whether there are any values $k \geq 7$, such that $I_k = I_{k+1}$ and in general, whether there are distinct values m, n, $m > n \geq 7$ such that $I_m = I_n$. He conjectured that there are none, but was not able to come up with a proof.

2453. [1999: 307, 428] *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^3}{(2n+1)!} \;=\; -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\,.$$

(b) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^3}{(2n)!} \;=\; -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\,.$$

(c) $$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!}\right)^2 \;=\; 2\,.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

(a) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^3}{(2n+1)!} \;=\; \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{2n(2n-1)}{(2n)!} + 3\sum_{n=0}^{\infty}(-1)^n\frac{2n}{(2n)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!} + 3\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!} + 3\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$

$$= -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!} \;=\; -3\sin 1\,.$$

(b) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^3}{(2n)!} \;=\; \sum_{n=1}^{\infty}(-1)^n\frac{(2n)^2}{(2n-1)!}$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)(2n-2)}{(2n-1)!} + 3\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)}{(2n-1)!} + \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}$$

$$= \sum_{n=2}^{\infty}(-1)^n\frac{1}{(2n-3)!} + 3\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!} + \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)!} + 3\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!} + \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}$$

$$= -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!} = -3\cos 1\,.$$

(c) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{(2n)!}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2n}{(2n)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= -\sin 1 + \cos 1\,,$$

and
$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{2n}{(2n-1)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{(2n-1)!} + \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}$$
$$= -\cos 1 - \sin 1\,.$$

Therefore,

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!}\right)^2$$
$$= (-\sin 1 + \cos 1)^2 + (-\cos 1 - \sin 1)^2$$
$$= 2\,.$$ ■

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NIKOLAOS DERGIADES, Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, WA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); and the proposer.

Janous pointed out that, as in Problem 2452, the identities in this problem all follow readily from results established in the proposer's paper Apropos Bell and Stirling Numbers. [*Ed: See comments at the end of the solution to Problem 2452.*] *More specifically,* (a) *holds if and only if* $c_3 \sin 1 + d_3 \cos 1 = -3(c_0 \sin 1 + d_0 \cos 1)$ *and* (b) *holds if and only if* $c_3 \cos 1 - d_3 \sin 1 = -3(c_0 \cos 1 - d_0 \sin 1)$. *These are true since* $c_0 = 1$, $c_3 = -3$ *and* $d_0 = d_3 = 0$. [*Ed: See Table 1 on p. 276 of the proposer's paper.*]

As for (c), *we have, more generally, that*

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^k}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^k}{(2n)!}\right)^2$$
$$= (c_k \sin 1 + d_k \cos 1)^2 + (c_k \cos 1 - d_k \sin 1)^2 = c_k^2 + d_k^2 .$$

[*Ed: In particular, for* $k = 2$ *we get the answer* 2 *since* $c_2 = -1$ *and* $d_2 = 1$. *For* $k = 3$ *and* 4 *we would get the answers* 9 *and* 61 *respectively. Hess commented that he determined these two values "from the misprinted problem"*].

2454. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three circles intersect each other orthogonally at pairs of points A and A', B and B', and C and C'. Prove that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

I. *Solution by Michel Bataille, Rouen, France.*

Among the three given circles, let Γ be the one that does not pass through A, and let I denote the inversion with centre A such that $I(\Gamma) = \Gamma$. Because of the mutual orthogonality of the three circles, I transforms the two other circles into two perpendicular lines that are orthogonal to $I(\Gamma) = \Gamma$.

Hence these two lines are diameters of Γ, meeting Γ at $I(B)$, $I(B')$, and $I(C)$, $I(C')$, respectively.

Clearly, the quadrilateral with vertices $I(B)$, $I(C)$, $I(B')$ and $I(C')$ is a rectangle, so that the lines through $I(B)$, $I(C)$, and through $I(B')$, $I(C')$, are parallel. Hence the images of these lines under I are circles, tangent at A. But these circles are precisely the circumcircles of $\triangle ABC$ and $\triangle AB'C'$, and so we are done.

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let the three circles be Γ_1, Γ_2 and Γ_3, and let A and A' be the intersections of Γ_1 and Γ_2, let B and B' be the intersections of Γ_1 and Γ_3, and let C and C' be the intersections of Γ_2 and Γ_3. See Figure 1.

We assume that Γ_3 intersects Γ_1 and Γ_2 orthogonally.

Let O_3 be the centre of Γ_3. Then O_3B and O_3B' are tangent to Γ_1 at B and B' respectively, and O_3C and O_3C' are tangent to Γ_2 at C and C' respectively. Note that O_3 lies on the line AA'.

Hence we have

$$\angle CAA' = \angle O_3CA' \quad \text{and} \quad \angle B'AA' = \angle O_3B'A' .$$

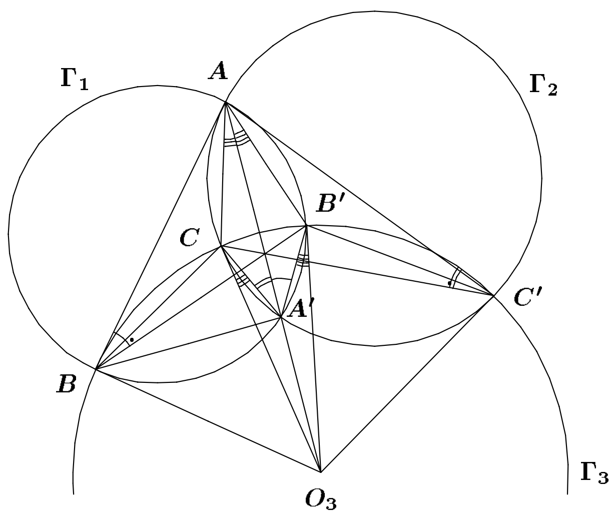


Figure 1.

It follows that

$$\begin{aligned}\angle CAB' &= \angle CAA' + \angle B'AA' = \angle O_3CA' + \angle O_3B'A' \\ &= \angle CA'B' - \angle CO_3B' . \qquad (1)\end{aligned}$$

Since O_3 is the centre of Γ_3, we have

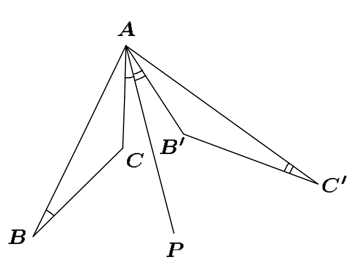
$$\angle CBB' = \angle CC'B = \tfrac{1}{2}\angle CO_3B' .$$

Since $\angle ABB' = \angle AA'B$ and $\angle AC'C = \angle AA'C$, we get

$$\begin{aligned}\angle ABC + \angle AC'B' &= (\angle ABB' - \angle CBB') + (\angle AC'C - \angle CC'B') \\ &= (\angle ABB' + \angle AC'C) - (\angle CBB' + \angle CC'B') \\ &= (\angle AA'B' + \angle AA'C) - \angle CO_3B' \\ &= \angle CA'B' - \angle CO_3B' .\end{aligned}$$

Thus we have, from (1),

$$\angle CAB' = \angle ABC + \angle AC'B' . \qquad (2)$$



Let P be an interior point of $\angle CAB'$ such that

$$\angle CAP = \angle ABC . \qquad (3)$$

Then we obtain from (2)

$$\angle B'AP = \angle AC'B' . \qquad (4)$$

From (3) and (4), we have that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ are tangent to AP at A. Thus, the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

Comment. As shown in the proof, the condition of the orthogonality of Γ_1 and Γ_2 is not necessary.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

All solvers other than Seimiya made use of inversion.

2455. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three equal circles, centred at A, B and C intersect at a common point P. The other intersection points are L (not on circle centre A), M (not on circle centre B), and N (not on circle centre C). Suppose that Q is the centroid of $\triangle LMN$, that R is the centroid of $\triangle ABC$, and that S is the circumcentre of $\triangle LMN$.

(a) Show that P, Q, R and S are collinear.

(b) Establish how they are distributed on the line.

Solution by Ho-joo Lee, Seoul, South Korea (modified by the editor).

Let $AB \bigcap PN = Z$, $BC \bigcap PL = X$, and $CA \bigcap PM = Y$. Since $\overline{AP} = \overline{AN} = \overline{BP} = \overline{BN}$, $PANB$ is a rhombus whose diagonals AB and PN are perpendicular and bisect one another (so that Z is the mid-point of both $\overline{AB}$ and $\overline{PN}$). Similarly for $BC \perp PL$ and $CA \perp PM$. Since $\triangle XYZ$ is the mid-point triangle of $\triangle ABC$ (with sides $XY||AB$, etc.), we have $ZP \perp XY$, $XP \perp YZ$, and $YP \perp ZX$, so that P is the orthocentre of $\triangle XYZ$. Moreover, R (the centroid of $\triangle ABC$) is the centroid of $\triangle XYZ$. Let T denote the circumcentre of $\triangle XYZ$. Thus P, R, and T lie on the Euler line of $\triangle XYZ$ and satisfy

$$\overline{PR} = 2\overline{RT}. \qquad (1)$$

The dilatation $P\left(\frac{1}{2}\right)$ not only carries $\triangle LMN$ to $\triangle XYZ$, but its centroid Q to R — so that

$$\overline{PQ} = 2\overline{PR}, \qquad (2)$$

and its circumcentre S to T — so that

$$\overline{QS} = 2\overline{RT}. \qquad (3)$$

Since P, R, and T are collinear, so are P, Q, R, and S. Finally, we have $\overline{PR} = \overline{RQ}$ (from (2)) and $\overline{PR} = \overline{QS}$ (from (1) and (3)), so that the points P, R, Q, and S are equally spaced in this order along the line.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; CHOONGYUP SUNG, Pusan, Korea; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

An immediate consequence of the featured proof is that P (the orthocentre of $\triangle XYZ$) is also the orthocentre of $\triangle LMN$, and that the circle LMN is congruent to the three given circles. Since $PLMN$ forms an orthocentric quadrangle (meaning each point is the orthocentre of the triangle formed by the other three), the configuration of the congruent circumcircles of these four triangles has a long history. The converse — that the circle LMN is congruent to the original 3 — can be traced back to R.A. Johnson [A circle theorem, American Math. Monthly **23** *(1916), 161-162;* see also *Arnold Emch,* Remarks on the foregoing circle theorem, **23** *(1916), 162-164], and to G. Ţiţeica (for whom the editor has never seen a specific reference, so does not know what he (or she) did and when it was done). This configuration was Dana McKenzie's starting point; his work* [Triquetras and porisms, *College Math. J.* **23** *2 (March, 1992), 118-131*] *would certainly be of interest to **CRUX with MAYHEM** readers.*

2456. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect orthogonally at P. A third circle touches them at Q and R. Let X be any point on this third circle. Prove that the circumcircles of $\triangle XPQ$ and $\triangle XPR$ intersect at $45°$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let P_1 be the second point of intersection of the circles which cut orthogonally.

Orthogonality invites inversion, so invert with respect to the circle with centre P and radius PP_1.

The orthogonal circles invert into a pair of mutually perpendicular lines through P_1, and the third circle becomes the circle tangent to these lines at the maps, Q' and R', of Q and P. That is to say, the inverse of the third circle is a circle on the chord $Q'R'$ which contains a right angle. Clearly, this circle passes through X', the inverse of the point X.

The circumcircles of $\triangle XOQ$ and $\triangle XPR$ become the straight lines through $X'Q'$ and $X'R'$ respectively.

Since an inscribed angle equals, in degrees, half of its intercepted arc, we clearly have

$$\angle Q'X'R' = \frac{90°}{2} = 45° \qquad \text{(Figure 1)}$$

or
$$\angle Q'X'R' = \frac{270°}{2} = 135° = 180° - 45° \qquad \text{(Figure 2)}$$

and the conclusion follows.

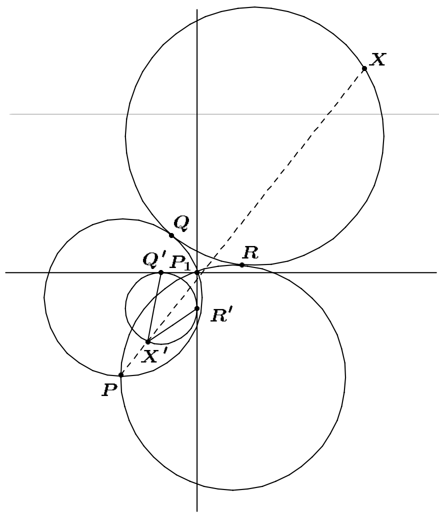


Figure 1

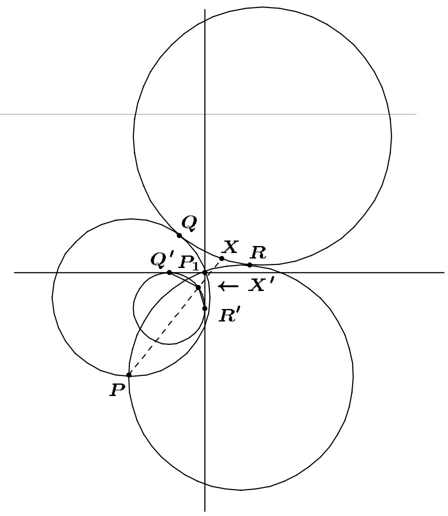


Figure 2

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

All solvers used inversion, except for Arslanagić, Seimiya and Smeenk, who used direct Euclidean methods.

2457. [1999: 308] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In quadrilateral $ABCD$, we have $\angle A + \angle B = 2\alpha < 180°$, and $BC = AD$. Construct isosceles triangles DCI, ACJ and DBK, where I, J and K are on the other side of CD from A, such that $\angle ICD = \angle IDC = \angle JAC = \angle JCA = \angle KDB = \angle KBD = \alpha$.

(a) Show that I, J and K are collinear.

(b) Establish how they are distributed on the line.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Since $\triangle ICD \sim \triangle JCA$, we have $\triangle JIC \sim \triangle ADC$ [by side-angle-side], so that $\angle JIC = \angle ADC$. Similarly, from $\triangle ICD \sim \triangle KBD$, we have $\triangle DIK \sim \triangle DCB$ and $\angle DIK = \angle DCB$. Therefore, $\angle JIK = \angle JIC + \angle DIK - \angle CID = (\angle ADC + \angle DCB) - \angle CID = (2\pi - 2\alpha) - (\pi - 2\alpha) = \pi$ [since in $ABCD$, $\angle A + \angle B = 2\alpha$, while $\triangle CID$ has base angles equal to α]. Consequently, JIK is a straight line.

Moreover, $\dfrac{IJ}{IC} = \dfrac{AD}{CD} = \dfrac{BC}{CD} = \dfrac{IK}{ID} = \dfrac{IK}{IC}$. Therefore, $IJ = IK$, or I is the mid-point of JK. *QED.*

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let M be the matrix that represents the rotation about the origin through angle $180^\circ - 2\alpha \neq 0$. So

$$M\cdot \overrightarrow{JA} = \overrightarrow{JC}\ , \quad M\cdot \overrightarrow{ID} = \overrightarrow{IC} \quad M\cdot \overrightarrow{KD} = \overrightarrow{KB}\ , \quad M\cdot \overrightarrow{DA} = \overrightarrow{CB}\ ,$$

and hence

$$\begin{aligned} M\cdot \overrightarrow{IJ} &= M\cdot\left(\overrightarrow{ID} + \overrightarrow{DA} + \overrightarrow{AJ}\right) = M\cdot \overrightarrow{ID} + M\cdot \overrightarrow{DA} - M\cdot \overrightarrow{JA} \\ &= \overrightarrow{IC} + \overrightarrow{CB} - \overrightarrow{JC} = \overrightarrow{IB} - \left(\overrightarrow{IC} - \overrightarrow{IJ}\right) \\ &= \overrightarrow{IB} - \overrightarrow{IC} + \overrightarrow{IJ}\ . \\ M\cdot \overrightarrow{IK} &= M\cdot\left(\overrightarrow{ID} + \overrightarrow{DK}\right) = M\cdot \overrightarrow{ID} - M\cdot \overrightarrow{KD} = \overrightarrow{IC} - \overrightarrow{KB} \\ &= \overrightarrow{IC} - \overrightarrow{IB} + \overrightarrow{IK}\ . \end{aligned}$$

By addition, $M\cdot\left(\overrightarrow{IJ} + \overrightarrow{IK}\right) = \overrightarrow{IJ} + \overrightarrow{IK}$, which implies that $\overrightarrow{IJ} + \overrightarrow{IK} = \overrightarrow{0}$, which means that the points I, J, K are collinear and that I is the mid-point of the segment JK.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece (a second solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2458. [1999: 308] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $ABCD$ be a quadrilateral inscribed in the circle centre O, radius R, and let E be the point of intersection of the diagonals AC and BD. Let P be any point on the line segment OE and let K, L, M, N be the projections of P on AB, BC, CD, DA respectively.

Prove that the lines KL, MN, AC are either parallel or concurrent.

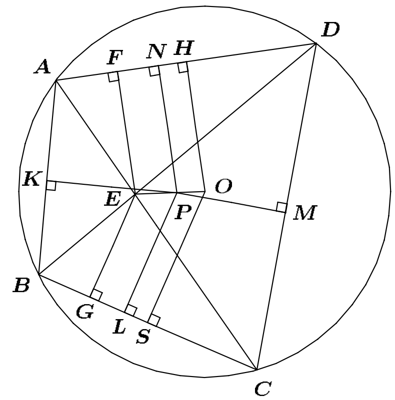
Solution by Toshio Seimiya, Kawasaki, Japan.

Let F and G be the feet of the perpendiculars from E to AD and BC, respectively. The triangles EAD and EBC are similar, because

$$\angle EAD = \angle CAD = \angle DBC = \angle EBC$$

and

$$\angle AED = \angle BEC\ .$$



Since $EF \perp AD$ and $EG \perp BC$, then $\triangle EAF$ is similar to $\triangle EBG$ and $\triangle EDF$ is similar to $\triangle ECG$. Hence $AF : BG = EF : EG = FD : GC$ and therefore,

$$AF : FD \;=\; BG : GC\,. \tag{1}$$

Let H and S be the feet of the perpendiculars from O to AD and BC, respectively. Since O is the centre of the circle, $AH = HD$ and $BS = SC$, giving

$$AH : HD \;=\; BS : SC\,. \tag{2}$$

Since EF, PN and OH are perpendicular to AD, then $EF \parallel PN \parallel OH$, and therefore, $FN : NH = EP : PO$. Similarly, $GL : LS = EP : PO$, so that

$$FN : NH \;=\; GL : LS\,. \tag{3}$$

From (1), (2) and (3) we obtain

$$AN : ND \;=\; BL : LC\,; \tag{4}$$

denote these two ratios by p. Similarly,

$$AK : KB \;=\; DM : MC\,; \tag{5}$$

denote these two ratios by q.

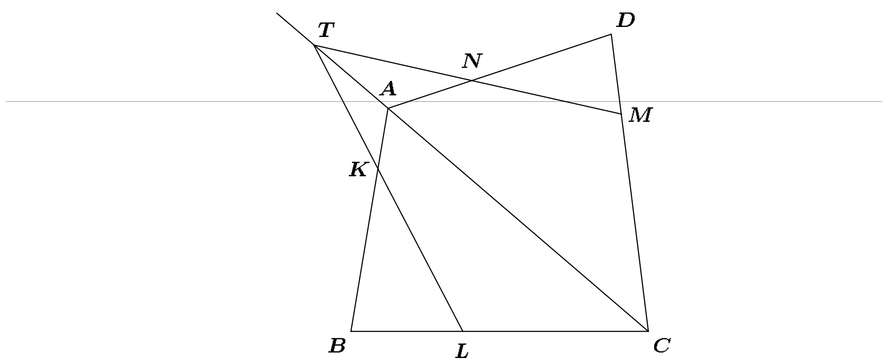
If $p = q$, then, by Thales' Theorem, $KL \parallel AC$ and $MN \parallel AC$.

If $p \neq q$, let T be the intersection of KL and AC. By Menelaus' Theorem,

$$\frac{AT}{TC} \cdot \frac{CL}{LB} \cdot \frac{BK}{KA} \;=\; 1\,.$$

Using (4) and (5), we obtain

$$\frac{AT}{TC} \cdot \frac{DN}{NA} \cdot \frac{CM}{MD} \;=\; 1\,.$$



It follows (by the converse of Menelaus' Theorem) that T, M and N are collinear. Therefore, KL, MN and AC are concurrent at T.

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2459. [1999: 308] *Proposed by Vedula N. Murty, Visakhapatnam, India, modified by the editors.*

Let P be a point on the curve whose equation is $y = x^2$. Suppose that the normal to the curve at P meets the curve again at Q. Determine the minimal length of the line segment PQ.

Solution by Michel Bataille, Rouen, France.

Let $P = (t, t^2)$ be a point on the parabola with $t \neq 0$. Then $Q = (u, u^2)$, where $u \neq 0$ and $u \neq t$. The slope of the tangent line at P is $2t$, so the slope of the normal at P is $-\frac{1}{2t}$; that is, $\frac{u^2-t^2}{u-t} = -\frac{1}{2t}$. Hence, $u = -t - \frac{1}{2t}$, and

$$\begin{aligned} PQ^2 &= (t-u)^2 + (t^2-u^2)^2 \;=\; (t-u)^2[1+(t+u)^2] \\ &= \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right) \;=\; \left(4t^2 + \frac{1}{4t^2} + 2\right)\left(1 + \frac{1}{4t^2}\right). \end{aligned}$$

Substituting $z = \frac{1}{4t^2} > 0$, we obtain

$$\begin{aligned} PQ^2 &= \left(z^2 + \frac{1}{z^2} + 2\right)\left(1 + \frac{1}{z^2}\right) \\ &= z^2 + 3z + 3 + \frac{1}{z} \;=\; \left(z - \frac{1}{2}\right)^2 \left(1 + \frac{4}{z}\right) + \frac{27}{4}. \end{aligned}$$

Hence $PQ^2 \geq \frac{27}{4}$ with equality if and only if $z = \frac{1}{2}$. Therefore, the minimal length of the segment PQ is $\frac{3\sqrt{3}}{2}$ obtained when P is either $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ or $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ELSIE CAMPBELL, Angelo State University, San Angelo, TX, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark (two solutions); NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; KARTHIK GOPALRATNAM, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; DAVID VELLA, Skidmore College, Saratoga Springs, NY, USA; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer. There was also one incorrect solution submitted.

Klamkin noted that this problem has appeared as # 11 in the first William Lowell Putnam Mathematical Competition [1]. Most of the solvers used Calculus to find the minimum. Dergiades, Leversha and Sung gave "No Calculus" solutions.

[1] *A.M. Gleason, R.E. Greenwood and L.M. Kelly, The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1938-1964, M.A.A., Washington, D.C., 1980, pp. 5, 91.*

Congratulations, Paco Bellot!

The International Federation of Mathematics Competitions has awarded the Paul Erdős Prize to Francisco Bellot Rosado in recognition of his work, since 1988, with Olympiad students. The prize will be presented in August 2000 during the International Conference of Mathematics Education in Japan. Francisco Bellot Rosado is the first Spaniard to receive this prize, and is the first High School teacher in the world to have been awarded it.

Congratulations, Paco Bellot (as he is known to his students and colleagues.)