MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

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Shreds and Slices

On Logarithms

It's log! — *The Ren & Stimpy Show*

The logarithm is an important function in mathematics and has some obvious uses, but it appears in some unexpected places as well. We give a diverse list of such applications, but first, we review some definitions.

In the expression $y = b^x$, b is called the *base* and x the *exponent*. With base b fixed, y is a function of x. The inverse function is called the *logarithm*, denoted $\log_b$. Thus, $\log_b y$ is the real number x such that $y = b^x$. For example, $\log_{10} 100 = 2$ and $\log_2 16 = 4$, since $10^2 = 100$ and $2^4 = 16$. The only acceptable bases are $0 < b < 1$ and $b > 1$.

Since the logarithm is based on the exponent function, the main properties of the logarithm are based on the main properties of exponents:

$$b^0 = 1, \quad (1)$$
$$b^1 = b, \quad (2)$$
$$b^u b^v = b^{u+v}, \quad (3)$$
$$b^u / b^v = b^{u-v}, \quad (4)$$
$$(b^u)^c = b^{uc}. \quad (5)$$

The corresponding properties of logarithms are:

$$\log_b 1 = 0, \quad (6)$$
$$\log_b b = 1, \quad (7)$$

$$\begin{aligned}
\log_b x + \log_b y &= \log_b xy\,, && (8)\\
\log_b x - \log_b y &= \log_b x/y\,, && (9)\\
\log_b x^c &= c\log_b x\,. && (10)
\end{aligned}$$

Let us prove these results.

Properties (6) and (7) follow directly from (1) and (2).

Let $u = \log_b x$ and $v = \log_b y$, so $x = b^u$ and $y = b^v$. Then $xy = b^{u+v}$, so $u + v = \log_b xy$. Also, $x/y = b^{u-v}$, so $u - v = \log_b x/y$, proving (8) and (9). Also, $x^c = (b^u)^c = b^{uc}$. Thus $uc = c\log_b x = \log_b x^c$, proving (10).

There is an additional important property, sometimes called the *change of base* formula:

$$\log_b x = \frac{\log_a x}{\log_a b}\,. \tag{11}$$

Since $b^u = x$, $\log_a b^u = \log_a x$. But $\log_a b^u = u\log_a b$, so that $u = \log_b x = \log_a x/\log_a b$. This means that logarithms to different bases are proportional to each other (see Problem 1).

- Base 10 is known as the *common* logarithm, but the most important base is a constant known as $e \approx 2.71828$. Why this constant? Let

  $$f(x) = \int_1^x \frac{1}{t}\,dt\,.$$

  For those not familiar with calculus, you can think of $f(x)$ as being the area under the graph of $y = 1/t$ from $t = 1$ to $t = x$. It turns out that $f(1) = 0$ and $f(x^c) = cf(x)$ for all c and $x > 0$. This implies that f is a logarithm to some base b. We define e to be this base. Hence,

  $$\log_e x = \int_1^x \frac{1}{t}\,dt\,.$$

  Unless otherwise indicated, all following logarithms will now be to base e, called the *natural* logarithm. One formula for e is $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

- $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\,, \qquad -1 < x \le 1\,.$

  What does the formula become for $x = 1$? What happens as x approaches -1?

- For all $x \ge 0$, $\log(1 + x) \le x$.

- For large n,
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \approx \log n \,.$$
In fact, the difference approaches a constant:
$$\gamma := \lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \approx 0.57722\,,$$
called the *Euler-Mascheroni* constant. It is currently not even known whether γ is irrational or not.

- Stirling's Approximation:
$$\log n! \approx n \log n - n \,.$$

- For a positive integer N, the number of digits in N expressed in decimal notation is $\lceil \log_{10}(N+1) \rceil$.

- One of the fundamental questions we can ask about prime numbers is how many there are up to a certain number. Let $\pi(n)$ be the number of primes less than or equal to n, and graph $\pi(n)$ in relation to $n/\log n$:

n	$\pi(n)$	$n/\log n$	$\pi(n)/(n/\log n)$
10	4	4.34	0.92
100	25	21.71	1.15
1,000	168	144.76	1.16
10,000	1,229	1,085.74	1.13
100,000	9,592	8,685.89	1.10
1,000,000	78,498	72,382.41	1.08

The agreement, given the irregularity of primes, is good. In fact, the Prime Number Theorem states that the two functions $\pi(n)$ and $n/\log n$ are *asymptotic*; that is, they will continue to approach each other:
$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1 \,.$$

- We give one final application. In finance circles, there is a rule of thumb called the Rule of 72: If an investment grows at a rate of $r\%$ annually, then the number of years it takes for the investment to double is approximately $72/r$.

For example, suppose that we have a dollar in a bank account, which earns 3% annually, so $r = 3$. Assuming that there are no other transactions, at the same time next year, there will be $\$1.03$ in the account. A year after that there will be $\$1.03 \times 1.03 = \$1.06$ in the account, and in general, after n years, there will be 1.03^n in dollars. The time in years n for the account to double to 2 dollars is given by $1.03^n = 2$.

Taking the logarithm of both sides, $\log 1.03^n = n \log 1.03 = \log 2$, so $n = (\log 2)/(\log 1.03) = 23.44$. The Rule of 72 gives $n \approx 72/3 = 24$, which is fairly close.

In general, we wish to solve for n in the equation

$$\begin{aligned} \left(1 + \frac{r}{100}\right)^n &= 2 \\ \Longrightarrow \log\left(1 + \frac{r}{100}\right)^n &= n \log\left(1 + \frac{r}{100}\right) = \log 2 \\ \Longrightarrow n &= \frac{\log 2}{\log(1 + \frac{r}{100})}. \end{aligned}$$

By the power series above,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \approx x$$

for small values of x, so $\log(1 + r/100) \approx r/100$, and

$$n \approx \frac{\log 2}{\frac{r}{100}} = \frac{100 \log 2}{r} = \frac{69.17...}{r} \approx \frac{72}{r}.$$

We round $100 \log 2$ to 72, because 72 is divisible by many numbers, and so provides for an easier calculation. The approximation is reasonable for many r, as seen in the table below:

r	n	$72/r$
1	69.66	72
2	35.00	36
3	23.45	24
4	17.67	18
6	11.90	12
8	9.01	9
9	8.04	8
12	6.11	6

Problems

1. In a table, list $\log_e n$ and $\log_{10} n$ for various values of n. Show that we can multiply all the values in one list by a constant to obtain all the values in the other list.

2. Before calculators, there was a device called a slide rule which performed numerical calculations. It consists of two, ruled pieces of material which slide against each other. Find out how it works.

Area of a Quadrilateral

The following is a useful, but not too well-known, formula for the area of a quadrilateral:

Let $ABCD$ be a quadrilateral, with sides $a = AB$, $b = BC$, $c = CD$, and $d = DA$. Let K denote the area, and s the semi-perimeter $(a+b+c+d)/2$. Let A also denote the angle at vertex A, etc. Then

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right).$$

We derive the formula as follows.

By the Cosine Law,

$$BD^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C.$$

Therefore,

$$a^2 - b^2 - c^2 + d^2 = 2ad\cos A - 2bc\cos C,$$

implying that

$$\begin{aligned}(a^2 - b^2 - c^2 + d^2)^2 &= 4a^2d^2\cos^2 A - 8abcd\cos A\cos C \\ &\quad + 4b^2c^2\cos^2 C.\end{aligned}$$

Now, the area of $\triangle ABD$ is $\frac{1}{2}ad\sin A$, and the area of $\triangle BCD$ is $\frac{1}{2}bc\sin C$. Thus

$$K = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C.$$

Therefore

$$\begin{aligned}16K^2 &= 4a^2d^2\sin^2 A + 8abcd\sin A\sin C + 4b^2c^2\sin^2 C \\ &= 4a^2d^2\sin^2 A + 8abcd\sin A\sin C + 4b^2c^2\sin^2 C \\ &\quad + 4a^2d^2\cos^2 A - 8abcd\cos A\cos C + 4b^2c^2\cos^2 C \\ &\quad - (a^2 - b^2 - c^2 + d^2)^2 \\ &= 4a^2d^2 + 4b^2c^2 - 8abcd\cos(A+C) - (a^2 - b^2 - c^2 + d^2)^2 \\ &= 4a^2d^2 + 8abcd + 4b^2c^2 - 8abcd\,(\cos(A+C) + 1) \\ &\qquad - (a^2 - b^2 - c^2 + d^2)^2 \\ &= (2ad + 2bc)^2 - (a^2 - b^2 - c^2 + d^2)^2 \\ &\qquad - 16abcd\cos^2\left(\frac{A+C}{2}\right) \\ &= (a^2 + 2ad + d^2 - b^2 + 2bc - c^2) \\ &\quad \times (b^2 + 2bc + c^2 - a^2 + 2ad - d^2) - 16abcd\cos^2\left(\frac{A+C}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left((a+d)^2-(b-c)^2\right)\left((b+c)^2-(a-d)^2\right) \\
&\qquad -16abcd\cos^2\left(\frac{A+C}{2}\right) \\
&= (-a+b+c+d)(a-b+c+d)(a+b-c+d) \\
&\qquad \times(a+b+c-d) \;-\; 16abcd\cos^2\left(\frac{A+C}{2}\right) \\
&= (2s-2a)(2s-2b)(2s-2c)(2s-2d) \\
&\qquad -16abcd\cos^2\left(\frac{A+C}{2}\right).
\end{aligned}
$$

Therefore

$$K^2 \;=\; (s-a)(s-b)(s-c)(s-d)-abcd\cos^2\left(\frac{A+C}{2}\right).$$

An immediate and important corollary of this result is Brahmagupta's Formula: If $ABCD$ is cyclic, then $A+C=180°$, so

$$K^2 \;=\; (s-a)(s-b)(s-c)(s-d).$$

You should now be able to quickly solve Problem 10 on the 1993 Descartes Competition:

Suppose p, q, r, s are fixed real numbers such that a quadrilateral can be formed with sides p, q, r, s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle.

Follow-up to "How to Solve the Cubic"

V.N. Murty writes to recommend the following textbooks as references for the cubic, as well as classic textbooks on algebra:

S. Barnard and J.M. Child, *Algebra*

G. Chrystal, *Algebra*, two volumes

M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 6 of 2001.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <ahchan@fas.harvard.edu>

H273. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, and c be complex numbers such that $a+b+c=0$. Prove that

$$\begin{vmatrix} 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \\ a^2 & c^2 & 2ac-b^2 \end{vmatrix} = 0\,.$$

H274. Find a simplified expression for

$$\sum_{i=1}^{\infty} \frac{i}{k^i}$$

in terms of a real number $k>1$.

H275. How many non-negative integers less than 10^n are there whose digits are in non-increasing order?

H276. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $ABCDE$ be a convex pentagon such that $ACDE$ is a square, and

$$\cot\angle BDE + \cot\angle DEB + \cot\angle EBD = 2\,.$$

Show that $\triangle ABC$ is a right triangle.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A249. *Proposed by Mohammed Aassila, Strasbourg, France.*
A circle is circumscribed around $\triangle ABC$ with sides a, b, c. Let A', B', C' denote the mid-points of the arcs BC, CA, AB, respectively. The straight lines $A'B'$, $B'C'$, $C'A'$ intersect BC and AC, AC and AB, AB and BC, in P, Q, R, S, T, U, respectively. Prove that

$$\frac{[PQRSTU]}{[ABC]} = \frac{(a+b)^2+(b+c)^2+(c+a)^2}{2(a+b+c)^2},$$

where $[X]$ denotes the area of the polygon X.

A250. Suppose polynomial $P(x)$ has integer coefficients such that for any integer m, $P(m)$ is a perfect square. Show that the degree of P is even.

A251. *Proposed by Lee Ho-Joo, undergraduate, Kwangwoon University, Seoul, South Korea.*
In a parallelogram $ABCD$, let P be the intersection of AC with BD. Let M, N be the mid-points of PD, BC, respectively. Prove that the following two statements are equivalent:

(i) $\triangle AMN$ is a (non-degenerate) right-angled triangle such that $AM = MN$.

(ii) Quadrilateral $ABCD$ is a square.

A252. *Proposed by Mohammed Aassila, Strasbourg, France.*
For every positive integer n, prove that there exists a polynomial of degree n with integer coefficients of absolute value at most n, which admits 1 as a root with multiplicity at least $\lfloor\sqrt{n}\rfloor$.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

In Issue 3, **C93** was accidentally printed as **C98**, and there was a typo in the problem. It is corrected here.

C93. Let H be a subset of the positive integers with the property that if x, $y \in H$, then $x + y \in H$. Define the *gap sequence* G_H of H to be the set of positive integers not contained in H.

(a) Prove that if G_H is a finite set, then the arithmetic mean of the integers in G_H is less than or equal to the number of elements in G_H.

(b) Determine all sets H for which equality holds in part (a).

C94. *Proposed by Edward Crane and Russell Mann, graduate students, Harvard University, Cambridge, MA, USA.*

Suppose that V is a k-dimensional vector subspace of the Euclidean space $\mathbb{R}^n$ which is defined by linear equations with coefficients in $\mathbb{Q}$. Let Λ be the lattice in V given by the intersection of V with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and let $\Lambda^\perp$ be the lattice given by the intersection of the perpendicular vector space $V^\perp$ with $\mathbb{Z}^n$. Show that the (k-dimensional) volume of Λ is equal to the (($n-k$)-dimensional) volume of $\Lambda^\perp$. [2000 : 167]

C95. Prove that the curve $x^3 + y^3 = 3xy$ has a horizontal tangent at the origin. (This curve is known as the Folium of Descartes.)

C96. Recall that a *bipartite graph* is a graph whose vertices may be divided into two nonempty disjoint sets (call them L and R, for left and right) so that all of the edges of the graph connect a vertex in L to a vertex in R. In other words, no two vertices in L are joined by an edge, and similarly for R. Let G be a bipartite graph with 27 edges and in which L and R each contain exactly 9 vertices. Show that we can find three vertices l_0, l_1, $l_2 \in L$ and three vertices r_0, r_1, $r_2 \in R$ such that at least six of the nine potential edges l_0r_0, l_0r_1, l_0r_2, l_1r_0, l_1r_1, l_1r_2, l_2r_0, l_2r_1, l_2r_2 are indeed edges of G.

1960 Bulgarian Mathematical Olympiad 1960

3rd Stage

1. Prove that the sum (difference) of two irreducible fractions with different denominators cannot be an integer.

2. Find the maximal and the minimal values of the function

$$y = \frac{x^2+x+1}{x^2+2x+2},$$

where x takes all real values.

3. Determine the tangents of x, y, z from the equations $\tan x : \tan y : \tan z = a : b : c$, if $x + y + z = 180^\circ$ and a, b, and c are positive numbers.

4. Two externally tangent circles of radii R and r are given.

(a) Prove that the quadrilateral whose sides are the two common tangents and the chords connecting the points of contact is a trapezoid.

(b) Find the base and the altitude of the trapezoid.

5. The rays a, b, and c have a common origin and do not lie in a plane. The angles $\alpha = \angle(b,c)$, $\beta = \angle(c,a)$, and $\gamma = \angle(a,b)$ are acute and are given in a plane. Construct by ruler and compass the angle between the ray a and the plane which passes through the rays b and c.

6. A sphere is inscribed in a cone. A second sphere tangent to the first is inscribed in the same cone. A third sphere tangent to the second is inscribed in the same cone and so on. Find the sum of the areas of the spheres if the altitude of the cone is equal to h and the angle at the vertex of its plane section through the axes is equal to α.

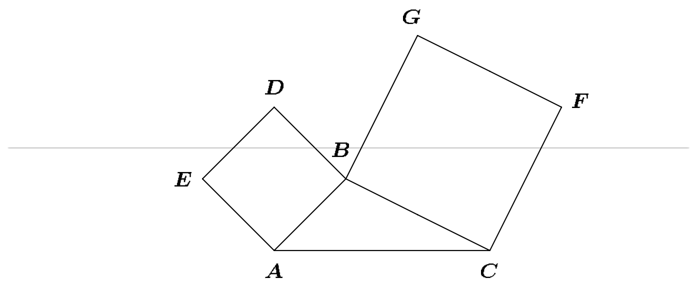
J.I.R. McKnight Problems Contest 1997

1. Given triangle ABC, with sides a, b, c, where $a+b+c = 60$ and $(b+c)/4 = (a+c)/5 = (a+b)/6$.
   (a) Prove that the sides a, b, c form an arithmetic sequence.
   (b) Find $\sin A : \sin B : \sin C$.
2. (a) Solve:
$$\frac{\log(35-x^3)}{\log(5-x)} = 3\,.$$
   (b) Prove:
$$\sum_{r=1}^{n} r\log_{2^r} x = n\log_2 x\,.$$
3. Consider three twin brothers: A, A, B, B, C, C. They are to be arranged in a picture in such a way, that no pair of twins will be side by side. Find the number of such arrangements.
4. The natural numbers are arranged in diamonds as shown below. Conjecture and prove a formula for the sum of the numbers in the n^{th} diamond.

$$\begin{array}{c}1\end{array}\qquad\begin{array}{ccc} &1& \\ 2&&2\\ &3& \end{array}\qquad\begin{array}{ccccc} &&1&& \\ &2&&2& \\ 3&&3&&3 \\ &4&&4& \\ &&5&& \end{array}\qquad\begin{array}{ccccccc} &&&1&&& \\ &&2&&2&& \\ &3&&3&&3& \\ 4&&4&&4&&4 \\ &5&&5&&5& \\ &&6&&6&& \\ &&&7&&& \end{array}$$

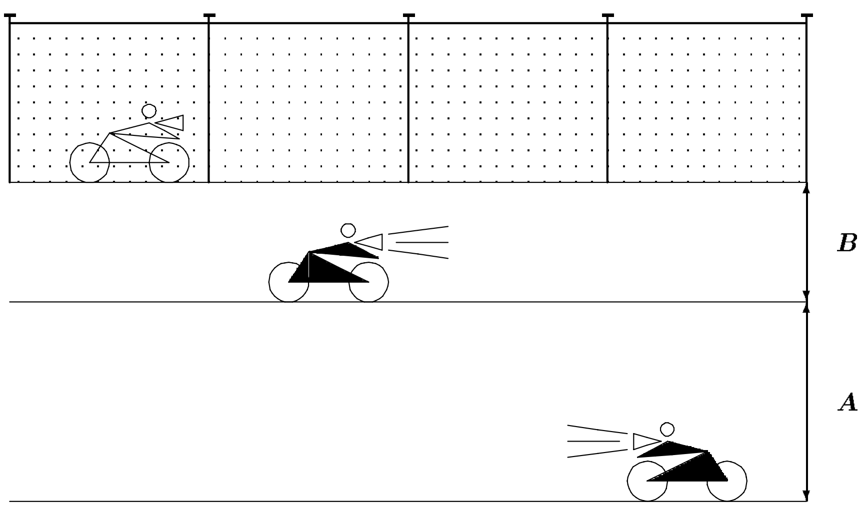
5. Find the limits N, M for which the inequality
$$N < \frac{(x+1)(x+4)}{x} < M$$
is satisfied by no real values of x.

6. Squares $ABDE$ and $BCFG$ are drawn outwards on the sides of $\triangle ABC$. Prove that AC is parallel to DG if and only if $\triangle ABC$ is isosceles.



7. Given $f(x+y)+f(x-y)=2f(x)\cos y$, $f(0)=a$, and $f(\pi/2)=b$. Find $f(t)$.

8. Two motorcycles are approaching each other at night on a straight, two-lane highway. Each vehicle is travelling in the centre of its lane and the centres of the 2 lanes are A metres apart. The eastbound cycle is travelling at M metres per second. The westbound cycle is travelling at a rate of N metres per second, and its headlight casts a shadow of the eastbound cycle onto a fence, B metres from the centre of the eastbound lane. How fast is the shadow of the eastbound cycle moving on the fence? (Express your answer in terms of A, B, M, and N).



Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Prove that

$$\frac{1}{1999} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998} < \frac{1}{44}\,.$$

(1997 CMO, Problem 3)

Solution. For the left inequality,

$$\begin{aligned}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998} &= \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{1997}{1996}\cdot\frac{1}{1998}\\ &> 1\cdot 1\cdot 1\cdots 1\cdot\frac{1}{1998}\\ &= \frac{1}{1998} > \frac{1}{1999}\,.\end{aligned}$$

For the right inequality, let

$$P = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998}\,.$$

Note that P is positive. Then,

$$\begin{aligned}P &< \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{1998}{1999}\\ &= \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{1998}{1997}\cdot\frac{1}{1999}\\ &= \frac{1}{1999P}\,,\end{aligned}$$

so that

$$P^2 < \frac{1}{1999} < \frac{1}{1936} = \frac{1}{44^2}\,,$$

which implies that $P < 1/44$. Thus,

$$\frac{1}{1999} < P < \frac{1}{44}\,.$$