

THE SKOLIAD CORNER

No. 47

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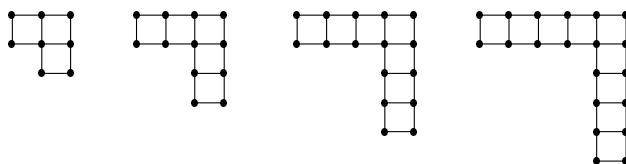
This issue we give the preliminary round of the Junior High School Mathematics Contest of the British Columbia Colleges. This was written in the schools by Grade 8 to 10 students on March 8, 2000. Thanks go to Jim Totten, The University College of the Cariboo, one of the organizers, for forwarding the contest materials to us.

BRITISH COLUMBIA COLLEGES Junior High School Mathematics Contest Preliminary Round — March 8, 2000

1. After 15 litres of gasoline was added to a partially filled fuel tank, the tank was 75% full. If the tank's capacity is 28 litres, then the number of litres in the tank before adding the gas was:

- (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

2. The following figures are made from matchsticks.



If you had 500 matchsticks, the number of squares in the largest such figure you could build would be:

- (a) 164 (b) 165 (c) 166 (d) 167 (e) none of these

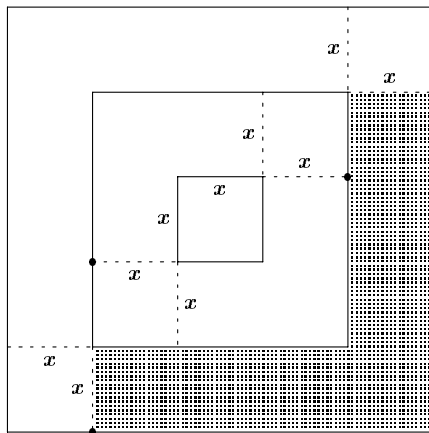
3. The perimeter of a rectangle is 56 metres. The ratio of its length to width is 4 : 3. The length, in metres, of a diagonal of the rectangle is:

- (a) 17.5 (b) 20 (c) 25 (d) 40 (e) none of these

4. If April 23 falls on Tuesday, then March 23 of the same year was a:

(a) Saturday (b) Sunday (c) Monday (d) Wednesday (e) Thursday

5. Consider the dart board shown in the diagram. If a dart may hit any point on the board with equal probability, the probability it will land in the shaded area is:

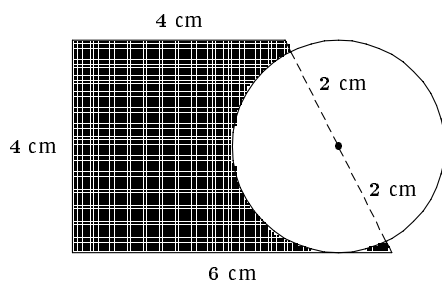


- (a) 0.07 (b) 0.24 (c) 0.25 (d) 0.28 (e) 0.32

6. The proper divisors of a number are those numbers that are factors of the number other than the number itself. For example, the proper divisors of 12 are 1, 2, 3, 4 and 6. An *abundant* number is defined as a number for which the sum of its proper divisors is greater than the number itself. For example, 12 is an abundant number since $1 + 2 + 3 + 4 + 6 > 12$. Another example of an abundant number is:

- (a) 13 (b) 16 (c) 30 (d) 44 (e) 50

7. The figure below is a right trapezoid with side lengths 4 cm, 4 cm, and 6 cm as labelled. The circle has radius 2 cm. The area, in cm^2 , of the shaded region is:

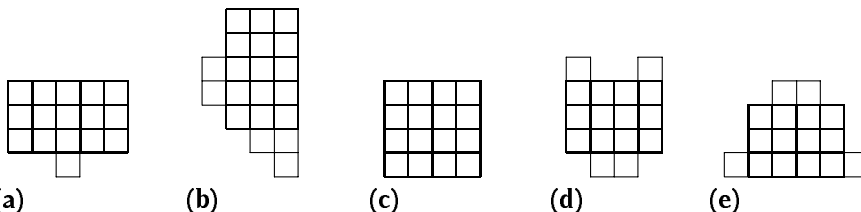


- (a) $20 - 4\pi$ (b) 16 (c) $24 - 2\pi$ (d) $20 - 2\pi$ (e) $16 + 2\pi$

8. Three vertices of parallelogram $PQRS$ were $P(-3, -2)$, $Q(1, -5)$, and $R(9, 1)$ with P and R diagonally opposite. The sum of the coordinates of vertex S is:

- (a) 13 (b) 12 (c) 11 (d) 10 (e) 9

9. Which shape *cannot* be filled, without any overlapping, using copies of the tile shown on the right?



10. Arrange the following in ascending order:

$$2^{5555} \quad 3^{3333} \quad 6^{2222}$$

- (a) 2^{5555} 3^{3333} 6^{2222} (b) 2^{5555} 6^{2222} 3^{3333}
 (c) 6^{2222} 3^{3333} 2^{5555} (d) 3^{3333} 6^{2222} 2^{5555}
 (e) 3^{3333} 2^{5555} 6^{2222}

11. 2000 days, 2000 hours, 2000 minutes, and 2000 seconds would be equivalent to N million seconds. Of the choices offered, the closest approximation of N is:

- (a) 1 (b) 15 (c) 45 (d) 180 (e) 2000

12. A three-digit decimal number abc may be expressed as $100a + 10b + c$ where each of the digits is multiplied by its respective place value and subsequently summed. If $a = b = c$ and $a > 0$, which of the following numbers must be a factor of the three digit number abc ?

- (a) 7 (b) 11 (c) 13 (d) 19 (e) 37

13. If $(x + y)^2 - (x - y)^2 > 0$, then

- (a) $(x > 0 \text{ and } y > 0)$ or $(x < 0 \text{ and } y < 0)$ (b) $x > 0$ and $y < 0$
 (c) $x < 0$ and $y > 0$ (d) $(x > 0 \text{ and } y < 0)$ or $(x < 0 \text{ and } y > 0)$
 (e) $x > y$ or $x < y$

14. Consider all non-congruent triangles with all sides having whole number lengths and a perimeter of 12 units. The following statements correspond to these triangles.

- (i) There are only three such triangles.
 (ii) The number of equilateral triangles equals the number of scalene triangles.
 (iii) None of these triangles are right angled.
 (iv) None of these triangles have a side of length 1 unit.

Of the four statements made, the number of true statements is:

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

15. An altitude, h , of a triangle is increased by a length m . How much must be taken from the corresponding base, b , so that the area of the new triangle is one-half that of the original?

(a) $\frac{bm}{h+m}$

(b) $\frac{bh}{2(h+m)}$

(c) $\frac{b(2m+h)}{m+h}$

(d) $\frac{b(m+h)}{2m+h}$

(e) $\frac{b(2m+h)}{2(h+m)}$

Last issue we gave the first round of two contests. Next we give short “official” solutions to the first of them. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Buenos Aires, for collecting them.

BUNDESWETTBEWERB MATHEMATIK
Federal Contest in Mathematics (Germany) 1997
First Round

1. Can you always choose 15 from 100 arbitrary integers so that the difference of any two of the chosen integers is divisible by 7?

What is the answer if 15 is replaced by 16? (Proof!)

Solution. The answer to the first question is *yes*. This is an elementary application of the Pigeon-Hole Principle, as one remainder (mod 7) must occur at least 15 times.

The answer to the second question is *no*. For a contradiction, take the integers from 1 to 100.

2. Determine all primes p for which the system

$$\begin{aligned} p + 1 &= 2x^2, \\ p^2 + 1 &= 2y^2, \end{aligned}$$

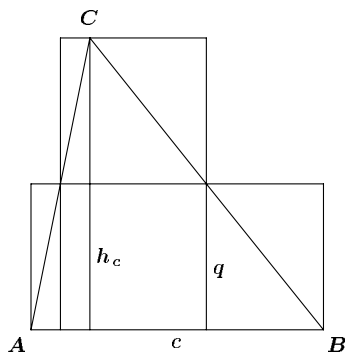
has a solution in integers x, y .

Solution. We can assume $y > x \geq 2$ and $p > y$. Subtracting the given equations, we have $p(p - 1) = 2(y - x)(y + x)$. It follows $p > y - x$ and $2p > y + x$, yielding $p = x + y$ and $p - 1 = 2(y - x)$. Eliminating y from the given equations, we get $p + 1 = 4x \implies x = 2 \implies p = 7$. Indeed, 7 satisfies the conditions.

3. A square S_a is inscribed in an acute-angled triangle ABC by placing two corners on the side BC and one corner on AC and AB , respectively. In a similar way, squares S_b and S_c are inscribed in ABC .

For which kind of triangle ABC do the sides of S_a, S_b and S_c have equal length?

Solution.



The inscribed squares are congruent if and only if triangle ABC is equilateral. The if-direction is trivial, so let us assume the squares to be congruent. Denoting the area of ABC by A and the length of the squares by q , we have $A = q^2 + \frac{1}{2}q(h_c - q) + \frac{1}{2}(c - q)q = \frac{1}{2}q(h_c + c)$. Cyclic permutation leads to $A = \frac{1}{2}q(h_b + b) = \frac{1}{2}q(h_a + a)$, thus $h_a + a = h_b + b$. With $2A = ah_a = bh_b$ we get $ah_a + a^2 = ah_b + ab$. Thus, $bh_b + a^2 = ah_b + ab$, and so, $(a - h_b)(a - b) = 0$. But $h_b < a$ for an acute-angled triangle, so $a = b$ and cyclic permutation leads to $a = b = c$.

4. In a park there are 10 000 trees, placed in a square lattice of 100 rows and 100 columns. Determine the maximum number of trees that can be cut down satisfying the condition:

sitting on a stump, you cannot see any other stump.

Solution. At most 2500 trees can be cut down and there is a way to chop exactly 2500 trees satisfying the conditions.

Let the trees have integer coordinates from $(0/0)$ to $(99/99)$. We divide the park into 2500 squares containing the trees $(2i/2j)$, $(2i/2j + 1)$, $(2i + 1/2j)$, $(2i + 1/2j + 1)$. Each tree in one of these squares is visible from the other lattice points in the square. Thus only one tree in each square can be cut down.

Let the 2500 trees with coordinates $(2i/2j)$ be cut down ($i, j \in \{0, 1, \dots, 49\}$). For any two lattice points $P(2a/2b)$ and $Q(2c/2d)$ ($a, b, c, d \in \{0, 1, \dots, 49\}$, $(a/b) \neq (c/d)$), the mid-point R of segment PQ has integer coordinates $(a + c/b + d)$. If one of these coordinates is odd, there is a tree left on R so that P and Q are invisible from each other. If both coordinates are even we can replace Q by R and repeat the same argument. After a finite number of steps we reach a tree. This completes the proof.

That completes the *Skoliad Corner* for this issue. We need good contest materials as well as suggestions for future directions.