Letter to the Editor

Regarding: Trevor Lipscombe & Arturo Sangalli; The Devil's dartboard [2000 : 215].

The arrangement of the numbers on a dartboard has intrigued the many players of the game and a number of mathematicians. I have outlined the history of the dartboard in my paper: Arranging a dartboard [*Bull. Inst. Math. Appl.* 16:4 (Apr 1980) 93-97; *CMP* 12:22 (1980) 1446; *MR* 81j:05005]. A simpler approach than used by Lipscombe & Sangalli is based on the idea that large and small numbers tend to alternate, so for a cyclic arrangement (a_i) of the first n integers, one could consider the sums of two adjacent numbers, $s_i = a_i + a_{i+1}$, and seek to make these as equal as possible; that is, have the least standard deviation or variance. This approach is much easier. It was used by Selkirk to determine the best and worst distributions, but the construction was a bit vague. I had also used the same approach and found an easy way to determine these optimal distributions. First one notes that minimizing the variance corresponds to minimizing the sum $C = \sum a_i a_{i+1}$. I considered a section $\ldots, a, b, \ldots, c, d, \ldots$ of the distribution and noted that reversing the portion from b to c reduced (or preserved) C unless $(a - d)(c - b) > 0$. But there is essentially only one distribution which satisfies this necessary condition and it looks like: $\ldots, 4, n - 2, 2, n, 1, n - 1, 3, n - 3, 5, \ldots$ I extended the analysis, relating it to the auto-correlation coefficient of the cycle with itself shifted by one and determining the mean and standard deviation of this auto–correlation, from which one can reasonably deduce that the designer of the standard dartboard must have had something like the idea of putting big numbers next to little ones in his mind.

I also considered making other sums as equal as possible, of which the most natural next stage is $s_i = a_{i-1} + a_i + a_{i+1}$, as considered by Lipscombe & Sangalli, and, more generally, $s_i = \sum p_j a_{i+j}$, where p_d is the probability of hitting the value which is d values from the one aimed at. Then the average of the s_i is the same as the average of the a_i, which is $\bar{a} = (n+1)/2$. Setting $D_d = \sum (a_i - \bar{a})(a_{i+d} - \bar{a})$, we have that D_d/nv is the auto–correlation coefficient of the cycle with itself shifted by d places — here v is the variance of the first n integers, namely $(n^2 - 1)/12$. Straightforward manipulation gives us an expression for the variance V of the s_i as $nV = \sum_{j,k} D_{k-j} p_j p_k$. We have $D_0 = nv$; $D_d = D_{-d} = D_{n-d}$ and $\sum D_i = 0$, which can be used to simplify the expression for nV. One usually also assumes symmetry of the p_d, but even so, the problem generally involves at least two D_d and different choices of the p_d will give different optima and a given set of probabilities may have several optima.

For the version considered by Lipscombe & Sangalli, we take $p_{-1} = p_0 = p_1 = 1/3$ and all other probabilities equal to 0, so we have $9nV = 3D_0 + 4D_1 + 2D_2$. For $n = 6$, the unique best distribution is $1, 6, 3, 2, 5, 4$, as also found by Lipscombe & Sangalli. However, for the simpler version considered above, corresponding to $p_0 = p_1 = 1/2$ and $4nV = 2D_0 + 2D_1$, the unique best distribution is $1, 6, 2, 4, 3, 5$. Returning to $p_{-1} = p_0 = p_1 = 1/3$, the case $n = 7$ has three best distributions: $1, 4, 7, 2, 3, 5, 6$; $1, 4, 7, 3, 2, 5, 6$; $1, 4, 7, 3, 2, 6, 5$. These are rather better than the distribution given by Lipscombe & Sangalli's algorithm, which I find is $7, 1, 4, 6, 2, 5, 3$. The technique of reversing a part of the distribution can be used here, but it leads to messy conditions which do not necessarily force a global minimum, though a computer could easily use them to improve an approximate minimum. The simplest case is reversing two adjacent terms in the arrangement; changing $\ldots, a, b, c, d, e, f, \ldots$ to $\ldots, a, b, d, c, e, f, \ldots$ decreases (preserves) the variance if $(a + b - e - f)(c - d) > 0 (= 0)$. For the result of Lipscombe & Sangalli, no such exchange reduces the variance, but exchanging 2 and 5 preserves the variance and in that arrangement, exchanging 1 and 4 does reduce the variance and gives a minimal arrangement.

References.

Keith Selkirk. Re-designing the dartboard. *Math. Gaz.* 60 (No.413) (1976) 171-178.

Ian Cook. Unbiased dartboards and biased calculators. *Math. Gaz.* 61 (No.417) (1977) 187-191. [He corrects and extends Selkirk, but considers different measures than treated here.]

David Daykin, proposer; David Singmaster, solver. Problem 1059: Cyclic extrema. *Math. Mag.* 52:1 (Jan 1979) 46 & 53:2 (Mar 1980) 115-116. [Asks for the extreme values of $\sum a_i a_{i+1}$ in a cycle of real values. See also *Crux Math.* 7:7 (Aug/Sep 1981) 210.]

Brian Bolt. **The Amazing Mathematical Amusement Arcade.** Cambridge Univ. Press, 1984. Prob. 116: Designing a new dartboard, pp. 67 & 123. [Asks to make $\sum |a_i - a_{i+1}|$ as large as possible. In the solution he poses finding the smallest value.]

Weixuan Li & Edward T.H. Wang, proposers; M.S. Klamkin & A. Meir, solvers. Problem E 3087 - Maximizing a cyclic sum of powers of differences. *Amer. Math. Monthly* 92 (1985) 287 & 94:4 (Apr 1987) 384-385. [Asks for the extreme values of $\sum (a_i - a_{i+1})^2$ where $0 \leq a_i \leq 1$. Solvers extend to $\sum |a_i - a_{i+1}|^p$.]

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