Some bounds for $\phi(n)\sigma(n)$

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Let $\mathbb{N}$ denote the set of all natural numbers. In elementary number theory, three multiplicative functions which are discussed most frequently are $\tau(n)$, the number of (positive) divisors of $n \in \mathbb{N}$; $\sigma(n)$, the sum of all the divisors of n; and $\phi(n)$, Euler's totient function; that is, the number of positive integers in $\{1, 2, \ldots, n\}$ which are coprime with n.

Though there are well-known formulae for computing the exact values of these functions, these formulae all depend on the actual prime power factorization of the natural number n. Hence it is of interest to find upper and lower bounds for these functions, preferably in terms of n only. Indeed, examples of such bounds abound in the literature. For example, the statements in the following proposition are clearly true and trivial.

Proposition 1. Let $n \in \mathbb{N}$, $n \geq 2$. Then

(a) $\tau(n) \geq 2$,

(b) $\phi(n) \leq n - 1$,

(c) $\sigma(n) \geq n + 1$.

In each of the three inequalities above, equality holds if and only if n is a prime.

On the other hand, there are less trivial bounds for these functions. For example, the inequalities in the next proposition can be found in ([1], p. 214 and p. 222).

Proposition 2. Let $n \in \mathbb{N}$. Then

(a) n is composite if and only if $\phi(n) < n - \sqrt{n}$,

(b) n is composite if and only if $\sigma(n) > n + \sqrt{n}$.

In view of Proposition 1, (b) and (c), it is natural to ask whether there is any inequality between $\phi(n)\sigma(n)$ and $n^2 - 1$. A quick numerical checking for $n \geq 2$ seems to suggest that $\phi(n)\sigma(n) \leq n^2 - 1$. In our first result (Proposition 4 below), we will show that this is indeed the case, but first, we need a lemma.

Lemma 3. Let $a_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$.
Then $(a_1 - 1)(a_2 - 1)\cdots(a_k - 1) \leq a_1 a_2 \cdots a_k - 1$. Equality holds if and only if either $k = 1$ or $a_i = 1$ for all i, $1 \leq i \leq k$.

Proof. We clearly have equality when $k = 1$. Hence we assume that $k \geq 2$. Let $a_i = 1 + b_i$ where $b_i \geq 0$ for all $i = 1, 2, \ldots, k$. Then

$$\left(\prod_{i=1}^{k} a_i\right) - 1 - \prod_{i=1}^{k}(a_i - 1) \;=\; \left(\prod_{i=1}^{k}(1 + b_i)\right) - 1 - \prod_{i=1}^{k} b_i \;=\; \sum_{m=1}^{k-1} S_m \;\geq\; 0\,,$$

where S_m is the m^{th} elementary symmetric function defined to be the sum of all $\binom{k}{m}$ possible products of the b_i's taken m at a time.

If equality holds, then $S_m = 0$ for all $m = 1, 2, \ldots, k-1$, from which it follows that $b_i = 0$ or $a_i = 1$ for all $i = 1, 2, \ldots, k$. This completes the proof. ∎

We now recall the familiar formulae for $\phi(n)$ and $\sigma(n)$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime power factorization of $n \in \mathbb{N}$, then

(a) $\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$,

(b) $\sigma(n) = \prod_{i=1}^{k} \left(1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i}\right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$.

Now we are ready for our first result.

Proposition 4. Let $n \in \mathbb{N}$, $n \geq 2$. Then $\phi(n)\sigma(n) \leq n^2 - 1$. Equality holds if and only if n is a prime.

Proof. First suppose n has more than one prime factor. Thus, let $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$ be the prime power factorization of n where $k \geq 2$. Since ϕ and σ are multiplicative functions, we have by Lemma 3, that

$$\begin{aligned}\phi(n)\sigma(n) \;&=\; \prod_{i=1}^{k}(\phi(p_i^{\alpha_i})\sigma(p_i^{\alpha_i})) \;\leq\; \prod_{i=1}^{k}\left(p_i^{2\alpha_i} - 1\right) \\ &<\; \left(\prod_{i=1}^{k} p_i^{2\alpha_i}\right) - 1 \;=\; n^2 - 1\,.\end{aligned}$$

Next suppose $n = p^\alpha$ for some prime p where $\alpha \in \mathbb{N}$. Then we have

$$\begin{aligned}\phi(n)\sigma(n) \;&=\; p^{\alpha-1}(p-1)\,\frac{p^{\alpha+1} - 1}{p - 1} \;=\; p^{\alpha-1}\left(p^{\alpha+1} - 1\right) \\ &=\; p^{2\alpha} - p^{\alpha-1} \;\leq\; p^{2\alpha} - 1 \;=\; n^2 - 1\end{aligned}$$

with equality if and only if $\alpha = 1$; that is, if and only if n is a prime. This completes the proof. ∎

Similarly, the two bounds in Proposition 2 raise the following natural question: Is there any inequality between $\phi(n)\sigma(n)$ and $n^2 - n$? Searching for a possible answer to this question we first do some computations and compare the values of $\phi(n)\sigma(n)$ with $n^2 - n$ for all n, $2 \le n \le 49$, as shown in the table below. Note that those integers n for which $\phi(n)\sigma(n) > n^2 - n$ are circled.

n	$\phi(n)\sigma(n)$	$n^2 - n$	n	$\phi(n)\sigma(n)$	$n^2 - n$	n	$\phi(n)\sigma(n)$	$n^2 - n$
②	3	2	18	234	306	34	864	1122
③	8	6	⑲	360	342	35	1152	1190
④	14	12	20	336	380	36	1092	1260
⑤	24	20	21	384	420	㊲	1368	1332
6	24	30	22	360	462	38	1080	1406
⑦	48	42	㉓	528	506	39	1344	1482
⑧	60	56	24	480	552	40	1440	1560
⑨	78	72	㉕	620	600	㊶	1680	1640
10	72	90	26	504	650	42	1152	1722
⑪	120	110	㉗	720	702	㊸	1848	1806
12	112	132	28	672	756	44	1680	1892
⑬	168	156	㉙	840	812	45	1872	1980
14	144	182	30	576	870	46	1584	2070
15	192	240	㉛	960	930	㊼	2208	2162
⑯	248	240	㉜	1008	992	48	1984	2256
⑰	288	272	33	960	1056	㊾	2394	2352

The above table seems to indicate that $\phi(n)\sigma(n)$ is greater than $n^2 - n$ ***exactly*** when n is a prime power. Our second result below shows that this is indeed the case.

Proposition 5. Let $n \in \mathbb{N}$, $n \ge 2$. Then $\phi(n)\sigma(n) > n^2 - n$ if and only if $n = p^k$ for some prime p where $k \in \mathbb{N}$.

Proof. The sufficiency is clear since if $n = p^k$, then

$$\phi(n)\sigma(n) = p^{2k} - p^{k-1} > p^{2k} - p^k = n^2 - n\,.$$

To prove the necessity, we show that if $n \neq p^k$, then in fact, $\phi(n)\sigma(n) < n^2 - n$. Assume first that n has only two distinct prime factors, so $n = p^\alpha q^\beta$ where α, $\beta \in \mathbb{N}$ and p and q are distinct primes. Then using the multiplicative property of ϕ and σ, we have

$$\phi(n)\sigma(n) = \phi(p^\alpha)\sigma(p^\alpha)\phi(q^\beta)\sigma(q^\beta) = \left(p^{2\alpha} - p^{\alpha-1}\right)\left(q^{2\beta} - q^{\beta-1}\right)$$

and so

$$\begin{aligned} n^2 - n - \phi(n)\sigma(n) &= p^{2\alpha}q^{2\beta} - p^\alpha q^\beta - \left(p^{2\alpha} - p^{\alpha-1}\right)\left(q^{2\beta} - q^{\beta-1}\right) \\ &= p^{2\alpha}q^{\beta-1} + p^{\alpha-1}q^{2\beta} - p^\alpha q^\beta - p^{\alpha-1}q^{\beta-1} \\ &= p^{\alpha-1}q^{\beta-1}(p^{\alpha+1} + q^{\beta+1} - pq - 1) \end{aligned}$$

$$
\begin{aligned}
&\geq p^{\alpha-1}q^{\beta-1}(p^2+q^2-pq-1)\\
&= p^{\alpha-1}q^{\beta-1}((p-q)^2+pq-1)\\
&> 0\,.
\end{aligned}
$$

Therefore, $\phi(n)\sigma(n) < n^2 - n$.

Now suppose $\phi(n)\sigma(n) < n^2 - n$ holds for all $n \in \mathbb{N}$ with t distinct prime factors for all $t = 2, 3, \ldots, m$ for some $m \geq 2$, and suppose $n \in \mathbb{N}$ has $m + 1$ distinct prime factors. Then we can write $n = p^k s$ where p is a prime and $s \in \mathbb{N}$ has m distinct prime factors and $(p, s) = 1$. Hence

$$
\begin{aligned}
\phi(n)\,\sigma(n) &= \phi(p^k)\sigma(p^k)\phi(s)\sigma(s)\\
&= \left(p^{2k}-p^{k-1}\right)\phi(s)\sigma(s)\\
&< \left(p^{2k}-p^{k-1}\right)(s^2-s)\,,
\end{aligned}
$$

where the inequality is by the induction hypothesis. Therefore

$$
\begin{aligned}
n^2-n-\phi(n)\sigma(n) &> p^{2k}s^2-p^ks-\left(p^{2k}-p^{k-1}\right)(s^2-s)\\
&= p^{2k}s+p^{k-1}s^2-p^ks-p^{k-1}s\\
&= p^{k-1}s(p^{k+1}-p+s-1)\\
&> 0\,,
\end{aligned}
$$

from which $\phi(n)\sigma(n) < n^2 - n$ follows and our induction is complete. ■

The corollary below clearly follows from Propositions 4 and 5:

Corollary 6. $\displaystyle\limsup_{n\to\infty} \frac{\phi(n)\sigma(n)}{n^2} = 1$.

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Reference

1. Kenneth H. Rosen, Elementary Number Theory and its Applications, 3rd ed., Addison-Wesley, 1993.

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