THE OLYMPIAD CORNER

No. 207

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I hope you spent the break working on problems and that I will soon be receiving your nice solutions!

We first give some Olympiad sets. We start with the Swedish Mathematical Competition, Final Round, November, 1996. My thanks go to Richard Nowakowski for collecting them when he was Canadian Team Leader at the IMO in Argentina.

SWEDISH MATHEMATICAL COMPETITION

Final Round

November 23, 1996 — Time: 5 hours

1. Through an arbitrary interior point of a triangle, lines parallel to the sides of the triangle are drawn dividing the triangle into six regions, three of which are triangles. Let the areas of these three triangles be T_1, T_2, and T_3 and let the area of the original triangle be T. Prove that

$$T = \left(\sqrt{T_1}+\sqrt{T_2}+\sqrt{T_3}\right)^2 .$$

2. In the country of Postonia, one wants to have only two values of stamps. The two values should be integers greater than one and the difference between the two should be two. It should also be possible to combine, in a precise way, stamps for each letter, the postage of which is greater than or equal to the sum of the two values. What values can be chosen?

3. For all integers $n \geq 1$ the functions p_n are defined for $x \geq 1$ by

$$p_n(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right) .$$

Show that $p_n(x) \geq 1$ and that $p_{mn}(x) = p_m(p_n(x))$.

4. A pentagon $ABCDE$ is inscribed in a circle. The angles at A, B, C, D, E form an increasing sequence. Show that the angle at C is greater than $\pi/2$. Also prove that this lower bound is best possible.

5. Let $n \geq 1$. Prove that it is possible to select some of the integers $1, 2, 3, \ldots, 2^n$ so that for all integers $p = 0, 1, \ldots, n-1$, the sum of k^p over all selected integers $k \in \{1, 2, 3, \ldots, 2^n\}$ is the same as the sum of k^p over all non-selected integers $k \in \{1, 2, 3, \ldots, 2^n\}$.

6. Tiles of dimension 6×1 are used to construct a rectangle. Prove that one of the sides has a length divisible by 6.

Next we give the problems of the 48th Polish Mathematical Olympiad, Final Round, written April 4–5, 1997. My thanks again go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina, and to Marcin E. Kuczma, Warszawa, Poland for sending me the problems.

48th POLISH MATHEMATICAL OLYMPIAD

Final Round – April 4–5, 1997

First Day — Time: 5 hours

1. The positive integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ satisfy the conditions:

$$x_6 = 144 \quad \text{and} \quad x_{n+3} = x_{n+2}(x_{n+1} + x_n) \quad \text{for} \quad n = 1, 2, 3, 4\,.$$

Compute x_7.

2. Solve the following system of equations in real numbers x, y, z:

$$\begin{aligned} 3(x^2 + y^2 + z^2) &= 1\,, \\ x^2y^2 + y^2z^2 + z^2x^2 &= xyz(x+y+z)^3\,. \end{aligned}$$

3. In a triangular pyramid $ABCD$, the medians of the lateral faces ABD, ACD, BCD, drawn from vertex D, form equal angles with the corresponding edges AB, AC, BC. Prove that the area of each lateral face is less than the sum of the areas of the two other lateral faces.

Second Day — Time: 5 hours

4. The sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 0\,, \qquad a_n = a_{[n/2]} + (-1)^{n(n+1)/2} \quad \text{for} \quad n > 1\,.$$

For every integer $k \geq 0$ find the number of all n such that

$$2^k \leq n < 2^{k+1}\,, \qquad a_n = 0$$

($[n/2]$ denotes the greatest integer not exceeding $n/2$).

5. Given is a convex pentagon $ABCDE$ with

$$DC = DE \quad \text{and} \quad \angle DCB = \angle DEA = 90^\circ .$$

Let F be the point on AB such that $AF : BF = AE : BC$. Show that

$$\angle FCE = \angle ADE \quad \text{and} \quad \angle FEC = \angle BDC .$$

6. Consider n points ($n \geq 2$) on the circumference of a circle of radius 1. Let q be the number of segments having those points as endpoints and having length greater than $\sqrt{2}$. Prove that $3q \leq n^2$.

Next we give the problems of the 18th Brazilian Mathematical Olympiad. Thanks again go to Richard Nowakowski for collecting and forwarding them to me when he was Canadian Team Leader to the IMO in Argentina.

18th BRAZILIAN MATHEMATICAL OLYMPIAD

1. Show that the equation

$$x^2 + y^2 + z^2 = 3xyz$$

has infinitely many integer solutions with $x > 0$, $y > 0$ and $z > 0$.

2. Is there a set A of n points ($n \geq 3$) in the plane such that:

(i) A does not contain three collinear points; and

(ii) given any three points in A, the centre of the circle which contains these points also belongs to A?

3. Let $f(n)$, $n \in \mathbb{Z}^+$, be the smallest number of ones that can be used to represent n using ones and any number of the symbols $+$, $\times$, (,), (with their usual meaning). For instance,

$$80 = (1+1+1+1+1) \times (1+1+1+1) \times (1+1+1+1)$$

and, therefore, $f(80) \leq 13$. Show that

$$3\log_3 n \leq f(n) < 5\log_3 n ,$$

for all $n > 1$. (Note: 11, 111, 1111, etc. may not be used in the expressions; only 1.)

4. Let D be a point of the side $\overline{BC}$ of the acute-angled triangle ABC ($D \neq B$ and $D \neq C$), O_1 be the circumcentre of $\triangle ABD$, O_2 be the circumcentre of $\triangle ACD$ and O be the circumcentre of $\triangle AO_1O_2$. Determine the locus described by the point O when D runs through the side $\overline{BC}$ ($D \neq B$ and $D \neq C$).

5. A set of marriages is *unstable* if two persons who are not married to each other prefer each other to their spouses. For instance, if Alessandra and Daniel are married and if Julia and Robinson are married, but Daniel prefers Julia to Alessandra, and Julia prefers Daniel to Robinson, then the set of marriages Alessandra-Daniel and Julia-Robinson is unstable. If the set of marriages is not unstable, we call it *stable*.

Consider now a group of people consisting of n boys and n girls. Each boy makes his own list ordering the n girls according to his preferences and, in the same way, each girl lists the n boys according to her preference. Show that it is always possible to marry the n boys and the n girls obtaining a stable marriage set.

6. Consider the polynomial $T(x) = x^3 + 14x^2 - 2x + 1$. Show that there exists a natural number $n > 1$ such that 101 divides $T^{(n)}(x) - x$ for all integers x. (Note: $T^{(n)}(x) = \underbrace{T(T(\cdots(T(x))))}_{n \text{ times}}$.)

As a fourth set to keep your solution skills finely honed, we give the problems from the Selection Test for the Vietnamese Team 1997, written May 16–17, 1997. Thanks again go to Richard Nowakowski who collected them for us when he was Canadian Team Leader to the IMO in Argentina.

SELECTION TEST FOR THE VIETNAMESE TEAM 1997

May 16–17, 1997

First Day — Time: 4 hours

1. Let $ABCD$ be a tetrahedron with $BC = a$, $CA = b$, $AB = c$, $DA = a_1$, $DB = b_1$, $CD = c_1$.

Prove that there exists one and only one point P satisfying the conditions:

$$\begin{aligned} PA^2 + a_1^2 + b^2 + c^2 &= PB^2 + b_1^2 + c^2 + a^2 \\ &= PC^2 + c_1^2 + a^2 + b^2 \;=\; PD^2 + a_1^2 + b_1^2 + c_1^2 \,, \end{aligned}$$

and that for this point P, we have $PA^2 + PB^2 + PC^2 + PD^2 \geq 4R^2$, where R is the radius of the circumscribed sphere of the tetrahedron $ABCD$. Find a necessary and sufficient condition on the lengths of the edges so that the preceding inequality becomes an equality.

2. In a country, there are 25 towns. Determine the least number k such that one can set up flight routes connecting these towns (in both directions) so that the following conditions are simultaneously satisfied:

(i) from each town there are exactly k direct flight routes to k other towns,

(ii) if there is no direct flight route connecting two towns, then there exists at least one town which has direct flight routes to these two towns.

3. Find the greatest real number α such that there exists an infinite sequence of whole numbers (a_n) $(n = 1, 2, 3, \dots)$ satisfying simultaneously the following conditions:

(i) $a_n > 1997^n$ for every $n \in \mathbb{N}^*$,

(ii) $a_n^\alpha \leq U_n$ for every $n \geq 2$, where U_n is the greatest common divisor of the set of numbers $\{a_i + a_j \mid i + j = n\}$.

Second Day — Time: 4 hours

4. Let $f : \mathbb{N} \to \mathbb{Z}$ be the function defined by:

$f(0) = 2$, $f(1) = 503$, $f(n+2) = 503f(n+1) - 1996f(n)$ for all $n \in \mathbb{N}$.

For every $k \in \mathbb{N}^*$, take k arbitrary integers $s_1, s_2, \dots, s_k$ such that $s_i \geq k$ for all $i = 1, 2, \dots, k$ and for every s_i $(i = 1, 2, \dots, k)$, take an arbitrary prime divisor $p(s_i)$ of $f(2^{s_i})$.

Prove that for positive integers $t \leq k$, we have:

$$\sum_{i=1}^{k} p(s_i) \mid 2^t \quad \text{if and only if} \quad k \mid 2^t \,.$$

5. Determine all pairs of positive real numbers a, b such that for every $n \in \mathbb{N}^*$ and for every real root x_n of the equation

$$4n^2 x = \log_2(2n^2 x + 1)$$

we have

$$a^{x_n} + b^{x_n} \geq 2 + 3x_n \,.$$

6. Let three positive integers n, k, p satisfying $k \geq 2$ and $k(p+1) \leq n$, be given.

Let n distinct points on a circle be given. One colours these n points blue and red (each point by a colour) so that there exist exactly k points coloured blue, and on each arc, the extremities of which are two consecutive (in clockwise direction) blue points, there exist at least p points coloured red.

What is the number of such colourings? (Two such colourings are distinct if there exists at least one point coloured with two different colours by these colourings).

Next we give solutions by our readers to problems given in the February 1999 number of the *Corner*. We start with solutions to problems of the Bundeswettbewerb Mathematik, Second Round 1995 [1999 : 4].

1. Starting in $(1, 1)$, a stone is moved according to the following rules:

(i) From any point (a, b), it can be moved to $(2a, b)$ or $(a, 2b)$.

(ii) From any point (a, b), if $a > b$ it can be moved to $(a - b, b)$, and if $a < b$ it can be moved to $(a, b - a)$.

Determine a necessary and sufficient relation between x and y so that the stone can reach (x, y) after some moves.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give the solution by Aassila.

Since $\gcd(1, 1) = 1$ and noting that $\gcd(p, q) = \gcd(p, q - p)$, we conclude that an odd common divisor can never be introduced. Hence $\gcd(x, y) = 2^r$ for some $r \in \mathbb{N}$ if (x, y) is reachable.

Assume now that $\gcd(x, y) = 2^r$ and let us prove that (x, y) is reachable. From the pairs (p, q) from which (x, y) can be reached, we choose the pair which minimizes $p + q$. In fact $p = q$; indeed if $p > q$ then (p, q) is reachable from $(\frac{p+q}{2}, q)$, a contradiction; similarly for the case in which $p < q$. Since $\gcd(p, q) = 2^r$ and neither p nor q is even, otherwise $p = q$ is not minimal for (p, q) from which (x, y) can be reached. We conclude that $p = q = 1$ and then (x, y) is reachable.

2. In a segment of unit length, a finite number of mutually disjoint subsegments are coloured such that no two points with distance 0.1 are both coloured. Prove that the total length of the coloured subsegments is not greater than 0.5.

Comment by Mohammed Aassila, Strasbourg, France. [*Ed: A solution was received from Pierre Bornsztein, Courdimanche, France.*]

This problem appeared as problem 6 of the Swedish Mathematical Competition 1986 and a solution appeared in [1992 : 296].

3. Every diagonal of a given pentagon is parallel to one side of the pentagon. Prove that the ratio of the lengths of a diagonal and its corresponding side is the same for each of the five pairs. Determine the value of this ratio.

Solutions by Mohammed Aassila, Strasbourg, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.

In the pentagon $ABCDE$, we assume that $AB \| CE$, $BC \| AD$, $CD \| BE$, $DE \| CA$, and $EA \| DB$. As shown in figure 1 on the next page, we label the intersections of diagonals.

Since $ATDE$ and $SCDE$ are both parallelograms we get:

$AT = ED = SC$, so that $AS + ST = ST + TC$. Thus we have $AS = TC$. Hence $CE : AB = CS : SA = AT : TC = AD : BC$.

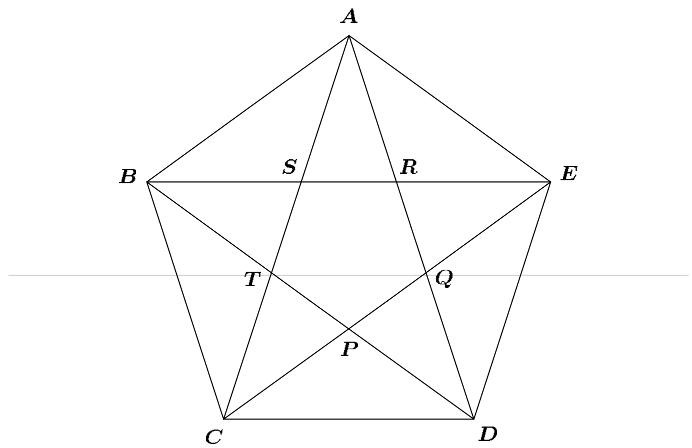


Figure 1.

Similarly we have

$$AD : BC \;=\; BE : CD \;=\; CA : DE \;=\; DB : EA\,.$$

We put

$$\frac{CE}{AB} \;=\; \frac{AD}{BC} \;=\; \frac{BE}{CD} \;=\; \frac{CA}{DE} \;=\; \frac{DB}{EA} \;=\; k\,.$$

Thus we have

$$\frac{CS}{SA} \;=\; \frac{CE}{AB} \;=\; k\,, \qquad \frac{DT}{BT} \;=\; \frac{AD}{BC} \;=\; k\,,$$

and $\dfrac{DR}{RA} = \dfrac{BD}{AE} = k$, so that $\dfrac{DB}{BT} = \dfrac{k+1}{1}$ and $\dfrac{TS}{SA} = \dfrac{k-1}{1}$.

By Menelaus' Theorem for $\triangle ATD$ we get

$$\frac{DB}{BT}\cdot\frac{TS}{SA}\cdot\frac{AR}{RD} \;=\; 1\,.$$

Therefore we have $\frac{k+1}{1}\cdot\frac{k-1}{1}\cdot\frac{1}{k} = 1$. Thus $k^2 - k - 1 = 0$, from which we obtain $k = \frac{1+\sqrt{5}}{2}$.

4. Prove that every integer k, $(k > 1)$ has a multiple which is less than k^4 and which can be written in decimal representation with at most four different digits.

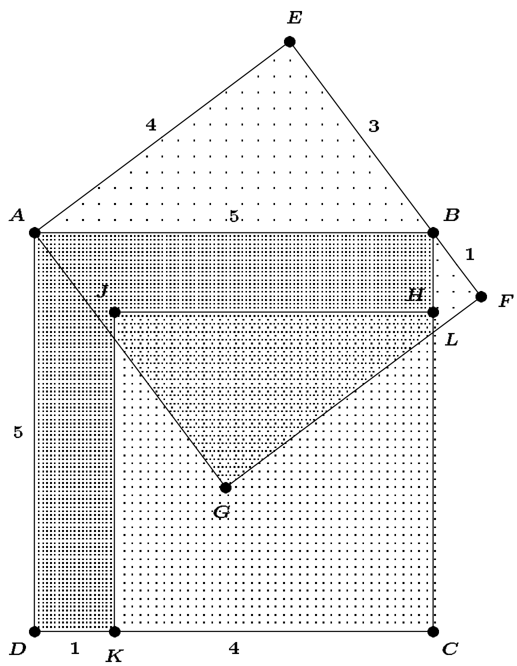
Comment by Mohammed Aassila, Strasbourg, France; and solution by Pierre Bornsztein, Courdimanche, France.

Aassila points out this problem was proposed by Poland but not used by the jury at the 31st IMO in China. A solution appeared in [1993 : 10].

Next we turn to the First Round 1996 of the Bundeswettbewerb Mathematik given in [1999 : 4].

1. Is it possible to cover a square of length 5 completely with three squares of length 4?

Solution by Sam Wong, student, Sir Winston Churchill High School, Calgary, Alberta.



This is a diagram showing the 5×5 square, $ABCD$, covered by two of the three available 4×4 squares. One 4×4 square shown is $JHCK$, and the other one is $AEFG$. The third 4×4 square is not shown, but occupies a similar position to square $AEFG$, except that it is reflected over a line joining AC. The reason that this square is not shown will be explained shortly.

The square $JHCK$ covers a 4×4 area of the square $ABCD$, and so leaves an L-shaped area (the polygon $ABHJKD$) left to be covered by the other two squares. To cover this L-shaped area with two squares, each square must cover, at the very least, half of this L-shaped area.

The square $AEFG$ does cover over half of the area. This can be proven as follows: Using Pythagoras' Theorem, the line EB can be calculated because it forms the triangle AEB and $AB = 5$, and $AE = 4$. Therefore, since $AB^2 = EB^2 + AE^2$, $EB^2 = 5^2 - 4^2$, or 9, so $EB = 3$.

On the left side of the diagram, $\angle BAG$ must be greater than $\angle BAC$. Since BAC is the diagonal of a square, $\angle BAC = 45°$. To find $\angle BAG$, we must first find $\angle EAB$ (since $\angle EAB$ and $\angle BAG$ are complementary). The sine of $\angle EAB$ is $\frac{3}{5}$, so when the inverse sine is taken for $\angle EAB$, the angle is $36.870°$. Subtract this angle from $90°$, and we get $(90° - 36.870°)$ or $53.130°$. Thus, $\angle BAG$ is approximately $53°$, and $53° > 45°$. Thus $\angle BAG$ is greater than $\angle BAC$.

On the right side of the diagram, point L must extend past point H on the line BC, or BL must be greater than BH (and BH is 1 unit long). Since BL is the hypotenuse of $\triangle BFL$, and BF is 1, then BL must be greater than 1. Thus BL is greater than BH.

When the third unit four square is placed as the reflection of $AEFG$ over the line AC, then this square will cover a similar area to $AEFG$. Both squares together will completely cover the aforementioned L-shaped area, and together with square $JHCK$, the unit five square is completely covered with three unit four squares.

2. The cells of an $n \times n$–board are numbered according to the example shown for $n = 5$. You may choose n cells, not more than one from each row and each column, and add the numbers in the cells chosen. Which are the possible values of this sum?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Solutions by Pierre Bornsztein, Courdimanche, France; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Sam Wong, student, Sir Winston Churchill High School, Calgary, Alberta. We give Wang's solution.

Using the terminology of combinatorial matrix theory, a collection of n entries from an $n \times n$ matrix $A = (a_{ij})$ with no two entries lying in the same row or the same column, is called a *diagonal* and the sum of the entries from a diagonal is called a *diagonal sum*. Note that if σ is a permutation of $\{1, 2, \ldots, n\}$ then $\{a_{i\sigma(i)} \mid i = 1, 2, \ldots, n\}$ would be a diagonal of A, and conversely, every diagonal gives rise to a permutation. Thus, A has exactly $n!$ diagonals.

We show that for the matrix considered, *all* diagonal sums equal $n(n^2 + 1)/2$. To see this, note that by assumption, $a_{ij} = (i - 1)n + j$ for all $i, j = 1, 2, \ldots, n$. Hence

$$\begin{aligned}\sum_{i=1}^{n} a_{i\sigma(i)} &= \sum_{i=1}^{n}((i-1)n+\sigma(i)) \\ &= \sum_{i=1}^{n}((i-1)n+i) \;=\; (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} n \\ &= \frac{n(n+1)^2}{2} - n^2 \;=\; \frac{n(n^2+1)}{2}\,.\end{aligned}$$

Remark. The sum of all the $n!$ diagonal sums is

$$S = \sum_{\sigma}\sum_{i=1}^{n} a_{i\sigma(i)} = n!\,\frac{n(n^2+1)}{2}\,.$$

On the other hand, since each entry of A lies on exactly $(n-1)!$ diagonals,

$$\begin{aligned}S \;=\; (n-1)!\sum_{i,j=1}^{n} a_{ij} &= (n-1)!\sum_{k=1}^{n^2} k \\ &= (n-1)!\,\frac{n^2(n^2+1)}{2} \;=\; n!\,\frac{n(n^2+1)}{2}\,,\end{aligned}$$

and so we have a check!

3. There are four straight lines in the plane, each three of them determining a triangle. One of these straight lines is parallel to one of the medians of the triangle formed by the other three lines. Prove that each of the other straight lines has the same property.

Solutions by Michel Bataille, Rouen, France; Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We first give Seimiya's geometric solution.

Let the four lines be a, b, c, d, and let $\triangle ABC$ be determined by a, b, and c as shown in the figure. The line d intersects BC, CA, AB at P, Q, R respectively.

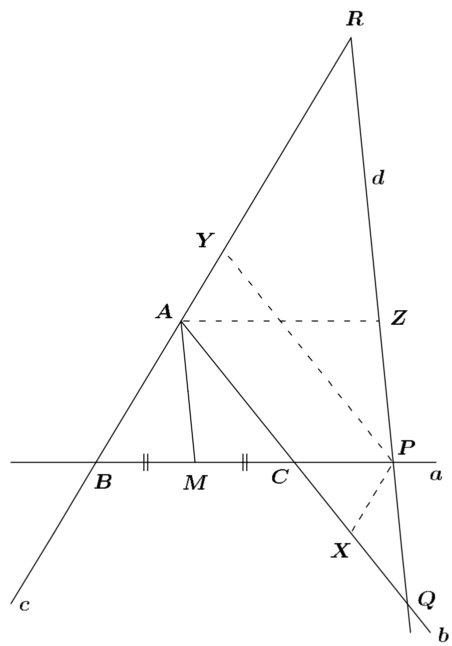
We assume that the median AM of $\triangle ABC$ is parallel to d. Let X be a point on AC such that $PX \| AB$. Since $PX \| AB$ and $AM \| PQ$, we have

$$\frac{AC}{CX} = \frac{BC}{CP} = \frac{2MC}{CP} = 2\cdot\frac{AC}{CQ}\,.$$

Thus $CQ = 2CX$; that is, $CX = XQ$.

Hence median PX of $\triangle PCQ$ is parallel to c. Let Y be a point on AB such that $PY \| AC$. Since $PY \| AC$ and $AM \| RP$, we have

$$\frac{BA}{BY} = \frac{BC}{BP} = \frac{2BM}{BP} = 2\cdot\frac{BA}{BR}\,.$$



Thus $BR = 2BY$; that is, $BY = YR$.

Hence median PY of $\triangle PBR$ is parallel to b. Let Z be a point on QR such that $AZ \| BP$. Since $AZ \| BP$ and $AM \| PQ$ we have

$$\frac{ZP}{ZQ} = \frac{AC}{AQ} = \frac{MC}{MP} = \frac{BM}{MP} = \frac{BA}{AR} = \frac{ZP}{ZR}.$$

Thus $ZQ = ZR$.

Hence median AZ of $\triangle AQR$ is parallel to a.

Next we give the solution of Bataille.

We denote the four straight lines by L_1, L_2, L_3, L_4, and we suppose that the names are chosen so that L_4 is parallel to one of the medians of the triangle formed by L_1, L_2, L_3. More precisely, let L_2, L_3 intersect at A, L_3, L_1 intersect at B, L_1, L_2 intersect at C and let us suppose $L_4 \| AM$ where M is the mid-point of BC. We shall work in the system of axes with origin A, and AB, AC as x–axis and y–axis respectively. Thus we have

$$A(0,0)\,; \quad B(1,0)\,; \quad C(0,1)\,; \quad M\left(\frac{1}{2},\frac{1}{2}\right)$$

and AM has equation $y = x$.

We readily find:

$$L_1 : x + y = 1\,; \quad L_2 : x = 0\,; \quad L_3 : y = 0\,, \quad \text{and } L_4 : x - y = k\,,$$

where $k \neq 0, 1, -1$ [this condition on k ensures that L_4 determines a real triangle with any two of the three lines L_1, L_2, L_3].

It is now easy to compute the coordinates of the points A', B', C' where L_4 intersects L_1, L_2, L_3 respectively:

$$A'\left(\frac{1+k}{2}, \frac{1-k}{2}\right); \quad B'(0,-k); \quad C'(k,0).$$

The mid-point I of BC' has coordinates $(\frac{1+k}{2}, 0)$ so that $\overrightarrow{A'I} = (0, \frac{k-1}{2})$. Hence the median $A'I$ (of the triangle formed by L_1, L_3, L_4) is parallel to L_2.

Similarly, the mid-point J of CB' has coordinates $(0, \frac{1-k}{2})$ and $\overrightarrow{A'J} = (-\frac{1+k}{2}, 0)$ is parallel to L_3.

Lastly, the mid-point K of $B'C'$ has coordinates $(\frac{k}{2}, -\frac{k}{2})$ and $\overrightarrow{AK} = (\frac{k}{2}, -\frac{k}{2})$ is parallel to L_1.

4. Determine the set of all positive integers n for which $n \cdot 2^{n-1}$ is a perfect square.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Let E be the set of such integers.

First Case. n is odd.

Write $n = 2k+1$ with $k \geq 0$. Then we have

$$n2^{n-1} = (2k+1)(2^k)^2 \quad \text{with } 2 \text{ and } 2k+1 \text{ coprime.}$$

Then $n \in E$ if and only if $2k+1 = n$ is a perfect square.

Second Case. n is even.

Write $n = 2^\alpha k$, $k \geq 1$ an odd integer and $\alpha \geq 1$. Then we have

$$n2^{n-1} = k2^{\alpha+2^\alpha k-1} \quad \text{with } k \text{ and } 2 \text{ coprime.}$$

Thus $n \in E$ if and only if $\alpha + 2^\alpha k - 1$ is even and k is a perfect square.

Thus $n \in E$ if and only if α is odd and k is a perfect square.

Further, $n \in E$ if and only if $n = 2b^2$ where $b \in \mathbb{N}^*$.

In summary, $n \in E$ if and only if n is an odd square or n is twice a square.

Now we turn to readers' comments and solutions to problems of the XLV Lithuanian Mathematical Olympiad 1996 given in [1999 : 5].

1. Solve the following equation in positive integers:

$$x^3 - y^3 = xy + 61 .$$

Solutions by Mohammed Aassila, Strasbourg, France; by Andrew Blinn, student, Western Canada High School, Calgary, Alberta; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Blinn's solution.

Manipulating the equation we obtain

$$(x-y)(x^2+xy+yz) - xy = 61$$

and

$$(x-y)((x-y)^2+3yx) - xy = 61 .$$

Set $x - y = a$. Since the right-hand side is positive, $a = x - y > 0$. Set $b = xy$. Note that $b > 0$.

Rewriting the equation in terms of a and b yields

$$a^3 + b(3a-1) = 61 ,$$

where $a > 0$ and $b > 0$.

Thus $a^3 < 61$, so $a = 1, 2, 3$.

Also $b = \frac{61-a^3}{3a-1} \in \mathbb{Z}$, so trying

$$\begin{array}{lll} a = 1 & \text{gives} & b = 30 , \\ a = 2 & \text{gives} & b = 53/5 , \\ a = 3 & \text{gives} & b = 17/4 . \end{array}$$

Thus $a = 1$, $x - y = 1$ and $xy = 30$, giving $(y+1)y = 30$ with $y = 5$ (rejecting $y = -6$) and $x = 6$.

The unique solution is $x = 6$, $y = 5$.

Comment. Aassila also points out that this problem was proposed at the 15^{th} All-Union Mathematical Olympiad held in Alma Ata, and a solution in Russian appears in N.B. Vassiliev and A.A. Egorov, "The Problems of the All-Union Mathematical Competitions", Moscow: Nauka, 1988.

2. Sequences $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ are such that $a_1 > 0$, $b_1 > 0$, and

$$a_{n+1} = a_n + \frac{1}{b_n} , \quad b_{n+1} = b_n + \frac{1}{a_n} , \quad n \in \mathbb{N} .$$

Prove that

$$a_{25} + b_{25} > 10\sqrt{2} .$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Heinz-Jürgen Seiffert, Berlin, Germany. We give the solution by Klamkin.

It follows that $a_2 + b_2 \geq 4$ and

$$a_{n+1} + b_{n+1} \;=\; (a_n + b_n)(1 + 1/a_n b_n) \;\geq\; (a_n + b_n) + 4/(a_n + b_n)\,.$$

Let $x_{n+1} = x_n + 4/x_n$ with $x_2 = 4$. Then since $x + 4/x$ is increasing for $x \geq 2$, we have $a_{n+1}+b_{n+1} \geq x_{n+1}$. Since $(x_{n+1})^2 = (x_n)^2 + (4/x_n)^2 + 8$,

$$(x_{n+1})^2 \;\leq\; (x_n)^2 + 8\,.$$

Summing the latter set of inequalities for $n = 2$ to $n - 1$, we obtain

$$(x_n)^2 \;\leq\; x_2^2 + 8(n-2) \;=\; 8n\,.$$

It then follows that

$$(x_{n+1})^2 \;\geq\; (x_n)^2 + 8 + 1/2n\,,$$

and summing this inequality for $n = 2$ to 24, we obtain

$$(x_{25})^2 \;\geq\; (x_2)^2 + 8(23) + (1/4 + 1/6 + \cdots + 1/48)\,.$$

Hence, $(x_{25})^2 > 16 + 184$ or $10\sqrt{2} < x_{25} \leq a_{25} + b_{25}$.

Next we give the generalization by Seiffert.

More generally: Let $u > 0$, $v > 0$, and $w > 0$. If the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ satisfy $a_1 > 0$, $b_1 > 0$, and

$$a_{n+1} \;=\; ua_n + \frac{v}{b_n}\,, \quad b_{n+1} \;=\; \frac{b_n}{u} + \frac{w}{a_n}\,, \quad n \in N\,,$$

then

$$a_n b_n \;>\; (n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}\,, \quad n \geq 3\,, \tag{1}$$

and

$$a_n + b_n \;>\; 2\sqrt{(n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}}\,, \quad n \geq 3\,. \tag{2}$$

First, we note that $a_n > 0$ and $b_n > 0$, $n \in \mathbb{N}$. We have

$$\begin{aligned} a_{k+1}b_{k+1} \;&=\; \left(ua_k + \frac{v}{b_k}\right)\left(\frac{b_k}{u} + \frac{w}{a_k}\right) \\ &=\; \frac{v}{u} + uw + a_k b_k + \frac{vw}{a_k b_k}\,, \quad k \in \mathbb{N}\,. \end{aligned}$$

Summing as k ranges from 1 to $n-1$, where $n \geq 3$, gives

$$\begin{aligned} a_n b_n &= (n-1)\left(\frac{v}{u} + uw\right) + a_1 b_1 + \sum_{k=1}^{n-1} \frac{vw}{a_k b_k} \\ &> (n-1)\left(\frac{v}{u} + uw\right) + a_1 b_1 + \frac{vw}{a_1 b_1} \\ &\geq (n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}\,, \end{aligned}$$

where we have used the AM-GM-Inequality. This proves (1). Then (2) follows from (1) and $a_n + b_n \geq 2\sqrt{a_n b_n}$.

In the particular case $u = v = w = 1$, (1) and (2) give

$$a_n b_n \;>\; 2n \quad \text{and} \quad a_n + b_n \;>\; 2\sqrt{2n}\,, \quad n \geq 3\,.$$

With $n = 25$, we then have $a_{25} b_{25} > 50$ and $a_{25} + b_{25} > 10\sqrt{2}$.

Remark. If $u = 1$ and $v = w$, then $a_1 b_n = b_1 a_n$ for all $n \in \mathbb{N}$, as is easily verified by induction on n.

3. Two pupils are playing the following game. In the system

$$\left\{ \begin{array}{l} *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \end{array} \right.$$

they alternately replace the asterisks by any numbers. The first player wins if the final system has a non-zero solution. Can the first player always win?

Solution by Pierre Bornsztein, Courdimanche, France.

Yes. Denote the system as follows:

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= 0\,, & (1) \\ b_1 x + b_2 y + b_3 z &= 0\,, & (2) \\ c_1 x + c_2 y + c_3 z &= 0\,. & (3) \end{aligned}$$

The first player chooses any number for b_2.

Then form the pairs (a_1, c_1), (a_2, c_2), (a_3, c_3), (b_1, b_3).

Each time the second player chooses a number from one pair, then the first player gives the same number to the other member of the pair. Thus at the end $a_1 = c_1$, $a_2 = c_2$, $a_3 = c_3$, $b_1 = b_3$. So (1) and (3) are equivalent.

And, since the system is homogeneous, it is consistent and must have infinitely many solutions, in particular a non-$(0,0,0)$ solution, and the first player wins.

4. How many sides has the polygon inscribed in a given circle and such that the sum of the squares of its sides is the largest one?

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's solution.

We will prove that the "maximal" polygon is the equilateral triangle.

Lemma. Let $\mathcal{P} = A_1A_2\ldots A_n$ be a convex polygon with $n \geq 5$ sides. Then $\mathcal{P}$ has an obtuse angle.

Proof of the lemma. We have $\sum_{i=1}^{n}\widehat{A_i} = (n-2)\pi$. Suppose, on the contrary, that each $\widehat{A_i} \leq \frac{\pi}{2}$. Then, $\sum_{i}^{n} A_i \leq \frac{n\pi}{2}$. Thus $(n-2)\pi \leq \frac{n\pi}{2}$. Then $n \leq 4$, which is a contradiction.

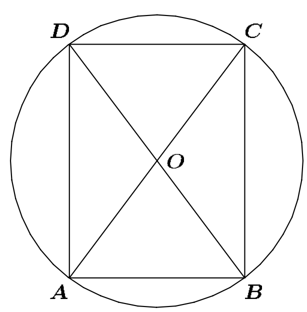
Let $\mathcal{P}$ be a convex polygon with $n \geq 4$ sides and with an obtuse angle. Let $\widehat{B} > \frac{\pi}{2}$ and A, C be the neighbouring vertices of B. Then, from the Law of Cosines:

$$AC^2 = AB^2 + BC^2 - 2AB\cdot BC\cos\widehat{B} > AB^2 + BC^2\,.$$

Thus, $\mathcal{P}$ is not maximal, because we have a larger sum by deleting B. Then, using the above lemma: if a "maximal" polygon exists, it has $n \leq 4$ sides, and no obtuse angle.

Let $\mathcal{P}_n$ be a convex polygon with $n \leq 4$ sides, no obtuse angle, inscribed in the circle $\mathcal{C}$ with centre O and radius R.

Let $n = 4$. The proof of the lemma may be used to prove that all angles of $\mathcal{P}_n$ are $\frac{\pi}{2}$ (because they are $\leq \frac{\pi}{2}$). Then $\mathcal{P}_4$ is a rectangle.



Pythagoras' Theorem leads to:

$$AB^2 + BC^2 = 4R^2 = CD^2 + DA^2\,.$$

Thus the sum of the squares of the sides is $S_4 = 8R^2$. Note that S_4 is independent of the rectangle.

Let $n = 3$. Suppose that $\triangle ABC$ is a non-obtuse triangle inscribed in $\mathcal{C}$. Let G be the centre of gravity of $\triangle ABC$. For any point M, we have

$$AM^2 + BM^2 + CM^2 = AG^2 + BG^2 + CG^2 + 3GM^2\,. \qquad (1)$$

Then

$$AG^2 + BG^2 + CG^2 + 3GO^2 \;=\; 3R^2 \,. \qquad (2)$$

Moreover, for $M = A$, we have $AB^2 + AC^2 = 4AG^2 + BG^2 + CG^2$. We also have the same relation for $M = B$, $M = C$.

Then, summing these three relations, we obtain

$$2(AB^2 + BC^2 + CA^2) \;=\; 6(AG^2 + BG^2 + CG^2)$$

and, using (2) we get

$$AB^2 + BC^2 + CA^2 \;=\; 9(R^2 - OG^2)\,.$$

We deduce that the sum of the squares is $S_2 \leq 9R^2$ with equality if and only if $O = G$; that is, $\triangle ABC$ is equilateral.

Then, the "maximal" polygon exists; it is the equilateral triangle and the sum is $S = 9R^2$.

5. Given ten numbers 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, one must cross out several of them so that the total of any of the remaining numbers would not be an exact square (that is, the sum of any two, three, four, ..., and of all the remaining numbers would not be an exact square). At most how many numbers can remain?

Solutions by Mohammed Aassila, Strasbourg, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The maximum number is five. First note that the numbers in each of the following sets add up to a perfect square: $\{2,7\}$, $\{3,6\}$, $\{3,13\}$, $\{5,11\}$, $\{6,10\}$, $\{12,13\}$, $\{3,10,12\}$. (And there are many more such sets, of course.) Let $S = \{2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13\}$ and partition S as $S = \{8\} \cup \{2,7\} \cup \{5,11\} \cup \{3,\ 6,\ 10,\ 12,\ 13\}$. Let $T \subseteq S$ be a set whose elements never add up to a perfect square and suppose $|T| \geq 6$. Then by the strong Pigeonhole Principle we have $|T \cap \{3,\ 6,\ 10,\ 12,\ 13\}| \geq 3$ which is impossible since if $3 \not\in T$, then either $\{6,10\} \subseteq T$ or $\{12,13\} \subseteq T$, a contradiction, and if $3 \in T$, then $6 \not\in T$ and $13 \not\in T$ would imply that $\{3,\ 10,\ 12\} \subseteq T$, again a contradiction. Therefore, $|T| \leq 5$. On the other hand, 5-element subsets of S whose elements do not add up to perfect squares do exist. One such set is $T = \{3,\ 7,\ 8,\ 11,\ 12\}$ since direct checkings show that the values of the sums of any k elements of T, $(1 \leq k \leq 5)$ are: 3, 7, 8, 10, 11, 12, 14, 15, 18, 19, 20, 21, 22, 23, 26, 27, 29, 30, 31, 33, 34, 38, and 41.

That completes the *Corner* for this issue. Send me your Olympiad Contest materials and your nice solutions to problems from the *Corner*.