

THE ACADEMY CORNER

No. 34

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Two items for your enjoyment this issue.

Memorial University Undergraduate Mathematics Competition

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- Find all roots of $(b - c)x^2 + (c - a)x + a - b = 0$ if a, b, c are in arithmetic progression (in the order listed).
- Evaluate $x^3 + y^3$ where $x + y = 1$ and $x^2 + y^2 = 2$.
- In triangle ABC , we have $\angle ABC = \angle ACB = 80^\circ$. P is chosen on line segment AB such that $\angle BPC = 30^\circ$. Prove that $AP = BC$.
- Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \dots + k(1999 - k) + \dots + (1997)(2) + (1998)(1)$.
- Let a_1, a_2, \dots, a_6 be 6 consecutive integers. Show that the set $\{a_1, a_2, \dots, a_6\}$ cannot be divided into two disjoint subsets so that the product of the members of one set is equal to the product of the members of the other. (Hint: First consider the case where one of the integers is divisible by 7.)
- Let $f(x) = x(x - 1)(x - 2) \dots (x - n)$.
 - Show that $f'(0) = (-1)^n n!$
 - More generally, show that if $0 \leq k \leq n$, then $f'(k) = (-1)^{n-k} k!(n - k)!$
- For each integer $n \geq 1$, let $\alpha_n = \sum_{j=1}^n 10^{-(j!)}$.
 - Show that $\lim_{n \rightarrow \infty} \alpha_n$ exists.
 - Show that $\lim_{n \rightarrow \infty} \alpha_n$ is irrational.

Send me your nice solutions!

The Bernoulli Trials 2000

Christopher G. Small & Byung Kyu Chun

Since 1997, the Bernoulli Trials, an undergraduate mathematics competition, has been held at the University of Waterloo. This is a double knockout competition. At the start of each round, students are presented with a mathematics statement which can be true or false. They have 10 minutes to determine the truth or falsehood of the proposition, and drop out after their second incorrect answer.

In March of 2000, there were 36 student participants. The competition lasted for 3.5 hours and 13 rounds, after which the first four places were clearly determined. The winner was **Scott Sitar**, who was the sole contestant not to be eliminated at the end of 12 rounds. Second place went to **Megan Davis**, who won a 13th round tie-breaker with 3rd place going to **Dennis The Adrian Tang** came in 4th, having survived to round 11. In keeping with the nature of the answers required, the prizes supplied by the Dean of Mathematics were awarded in coins: 200 dollars (100 “toonies”) for first, 100 dollars (“loonies”) for second, 70 dollars for third, and 30 dollars for fourth.

1. A deck of 2000 cards has the numbers from 1 to 2000 labelled consecutively in order from top to bottom. The deck is shuffled as follows. The second card from the top is placed on the top card, the third card is placed below these two, the fourth above these three, the fifth below these four, and so on, until the 2000th card is placed above the remaining 1999.

TRUE or FALSE? At the completion of this shuffle, every card is in a different position in the deck than where it started; that is, for every $i = 1, \dots, 2000$ the card labelled i is not in position i .

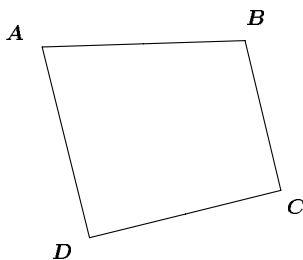
2. **TRUE OR FALSE?**

The equation

$$\sin(\sin(\sin(x))) = x/3$$

has exactly one solution in real values x .

3. Let $ABCD$ be a planar convex quadrilateral labelled clockwise as shown:



Suppose that

$$[ABC] \leq [BCD] \leq [CDA] \leq [DAB]$$

where $[RST]$ represents the area of triangle RST .

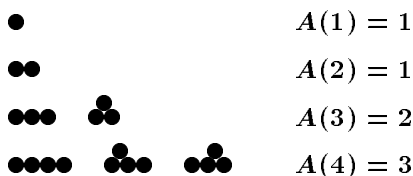
TRUE or FALSE? AD is parallel to BC .

(A quadrilateral is said to be *convex* if no vertex is within the triangle formed by the other three vertices.)

4. We arrange dimes in rows on top of each other according to the following rules:

- each coin must touch the next in its row;
- each coin except those in the bottom row touches two coins on the row below.

Let $A(n)$ be the number of distinct ways to arrange n coins. For example, $A(4) = 3$ as shown.



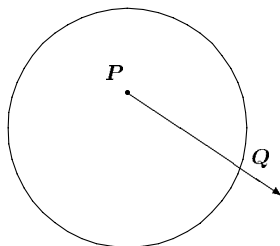
TRUE or FALSE? $A(n)$ is the n^{th} Fibonacci number; that is, $A(1) = 1$, $A(2) = 1$, \dots , $A(n+2) = A(n) + A(n+1)$.

5. **TRUE or FALSE?** For every integer $n \geq 3$, the equation

$$x^n + y^n = z^{n+1}$$

has infinitely many solutions in positive integers x , y and z .

6. Consider a point P at random inside a circle of diameter 2. From P , a ray is drawn in a random direction, and intersects the circumference of the circle at Q .



TRUE OR FALSE? The average length of PQ is 1.

7. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be any n vectors in \mathbb{R}^n such that $\|\mathbf{v}_i\| = 1$, for all $i = 1, \dots, n$.

TRUE OR FALSE? It is always possible to select $\epsilon_1, \dots, \epsilon_n \in \{-1, +1\}$, so that

$$\|\epsilon_1 \mathbf{v}_1 + \dots + \epsilon_n \mathbf{v}_n\| \geq \sqrt{n}$$

and $\delta_1, \dots, \delta_n \in \{-1, +1\}$, so that

$$\|\delta_1 \mathbf{v}_1 + \dots + \delta_n \mathbf{v}_n\| \leq \sqrt{n}.$$

8. **TRUE or FALSE?** For all $0 < x < 1$,

$$\frac{d^{2000}}{dx^{2000}} [\ln(x) \ln(1-x)] < 0.$$

9. Two players, Arthur and Barbara, take turns selecting numbers from the set

$$\{1, 2, 3, 4, \dots, 9\}.$$

A number, after selection, cannot be selected in a subsequent round. The first player to obtain a set of 3 numbers totalling 15 is the winner.

TRUE or FALSE? With best play by both sides, the first player (Arthur) can force a win.

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Suppose that for every rational number q there exists a positive integer N such that $f^n(q) = 0$ for all $n \geq N$, where f^n denotes the n -fold iteration of f .

TRUE or FALSE? For every real number t

$$\lim_{n \rightarrow \infty} f^n(t) = 0.$$

11. **TRUE or FALSE?**

$$\sum_{k=0}^{\infty} \frac{8k^3 + 4k + 1}{(2k)!} < \sum_{k=0}^{\infty} \frac{(2k+1)^3 + 4k + 3}{(2k+1)!}$$

12. Consider a sequence of positive integers

$$a_0, a_1, a_2, \dots$$

with the property that a_n equals the number of positive divisors of a_{n-1} . (The number a_i has both 1 and a_i as divisors.) We set $a_0 = 2000!$.

TRUE or FALSE? For some positive integer n the number a_n is a perfect square.

13. **TRUE or FALSE?** There exists a function $f : [-1, +1] \rightarrow \mathbb{R}$ with continuous second derivative such that

$$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right) \text{ converges and } \sum_{n=1}^{\infty} \left|f\left(\frac{1}{n}\right)\right| \text{ diverges.}$$