THE ACADEMY CORNER

No. 34

Bruce Shawyer

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Two items for your enjoyment this issue.

Memorial University Undergraduate Mathematics Competition

March 2000

1. Find all roots of $(b-c)x^2 + (c-a)x + a - b = 0$ if a, b, c are in arithmetic progression (in the order listed).
2. Evaluate $x^3 + y^3$ where $x + y = 1$ and $x^2 + y^2 = 2$.
3. In triangle ABC, we have $\angle ABC = \angle ACB = 80^\circ$. P is chosen on line segment AB such that $\angle BPC = 30^\circ$. Prove that $AP = BC$.
4. Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \ldots + k(1999-k) + \ldots + (1997)(2) + (1998)(1)$.
5. Let a_1, a_2, $\ldots$, a_6 be 6 consecutive integers. Show that the set $\{a_1, a_2, \ldots, a_6\}$ cannot be divided into two disjoint subsets so that the product of the members of one set is equal to the product of the members of the other. (Hint: First consider the case where one of the integers is divisible by 7.)
6. Let $f(x) = x(x-1)(x-2)\ldots(x-n)$.
   (a) Show that $f'(0) = (-1)^n n!$
   (b) More generally, show that if $0 \le k \le n$, then $f'(k) = (-1)^{n-k}k!(n-k)!$
7. For each integer $n \ge 1$, let $\alpha_n = \sum\limits_{j=1}^{n} 10^{-(j!)}$.
   (a) Show that $\lim\limits_{n\to\infty} \alpha_n$ exists.
   (b) Show that $\lim\limits_{n\to\infty} \alpha_n$ is irrational.

Send me your nice solutions!

The Bernoulli Trials 2000

Christopher G. Small & Byung Kyu Chun

Since 1997, the Bernoulli Trials, an undergraduate mathematics competition, has been held at the University of Waterloo. This is a double knockout competition. At the start of each round, students are presented with a mathematics statement which can be true or false. They have 10 minutes to determine the truth or falsehood of the proposition, and drop out after their second incorrect answer.

In March of 2000, there were 36 student participants. The competition lasted for 3.5 hours and 13 rounds, after which the first four places were clearly determined. The winner was **Scott Sitar**, who was the sole contestant not to be eliminated at the end of 12 rounds. Second place went to **Megan Davis**, who won a 13$^{\text{th}}$ round tie-breaker with 3$^{\text{rd}}$ place going to **Dennis The**. **Adrian Tang** came in 4$^{\text{th}}$, having survived to round 11. In keeping with the nature of the answers required, the prizes supplied by the Dean of Mathematics were awarded in coins: 200 dollars (100 "toonies") for first, 100 dollars ("loonies") for second, 70 dollars for third, and 30 dollars for fourth.

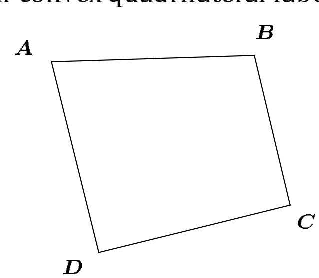
1. A deck of 2000 cards has the numbers from 1 to 2000 labelled consecutively in order from top to bottom. The deck is shuffled as follows. The second card from the top is placed on the top card, the third card is placed below these two, the fourth above these three, the fifth below these four, and so on, until the 2000th card is placed above the remaining 1999.

   TRUE or FALSE? At the completion of this shuffle, every card is in a different position in the deck than where it started; that is, for every $i = 1, \ldots, 2000$ the card labelled i is not in position i.

2. **TRUE OR FALSE?**

   The equation
   $$\sin(\sin(\sin(x))) = x/3$$
   has exactly one solution in real values x.

3. Let $ABCD$ be a planar convex quadrilateral labelled clockwise as shown:



Suppose that

$$[ABC] \leq [BCD] \leq [CDA] \leq [DAB]$$

where $[RST]$ represents the area of triangle RST.

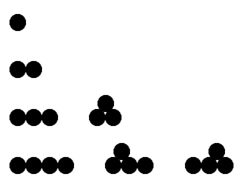
TRUE or FALSE? AD is parallel to BC.

(A quadrilateral is said to be *convex* if no vertex is within the triangle formed by the other three vertices.)

4. We arrange dimes in rows on top of each other according to the following rules:

   - each coin must touch the next in its row;
   - each coin except those in the bottom row touches two coins on the row below.

   Let $A(n)$ be the number of distinct ways to arrange n coins. For example, $A(4) = 3$ as shown.



$$A(1) = 1$$
$$A(2) = 1$$
$$A(3) = 2$$
$$A(4) = 3$$

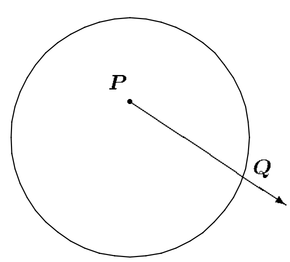
   TRUE or FALSE? $A(n)$ is the n^{th} Fibonacci number; that is, $A(1) = 1$, $A(2) = 1$, $\ldots$, $A(n+2) = A(n) + A(n+1)$.

5. **TRUE or FALSE?** For every integer $n \geq 3$, the equation

$$x^n + y^n = z^{n+1}$$

   has infinitely many solutions in positive integers x, y and z.

6. Consider a point P at random inside a circle of diameter 2. From P, a ray is drawn in a random direction, and intersects the circumference of the circle at Q.



   TRUE OR FALSE? The average length of PQ is 1.

7. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be any n vectors in $\mathbb{R}^n$ such that $||\mathbf{v}_i|| = 1$, for all $i = 1, \ldots, n$.

   TRUE OR FALSE? It is always possible to select $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$, so that
$$||\epsilon_1 \mathbf{v}_1 + \cdots + \epsilon_n \mathbf{v}_n|| \geq \sqrt{n}$$
   and $\delta_1, \ldots, \delta_n \in \{-1, +1\}$, so that
$$||\delta_1 \mathbf{v}_1 + \cdots + \delta_n \mathbf{v}_n|| \leq \sqrt{n}\,.$$

8. **TRUE or FALSE?** For all $0 < x < 1$,
$$\frac{d^{2000}}{dx^{2000}} [\ln(x) \ln(1-x)] < 0 \ .$$

9. Two players, Arthur and Barbara, take turns selecting numbers from the set
$$\{1, 2, 3, 4, \ldots, 9\}\,.$$
   A number, after selection, cannot be selected in a subsequent round. The first player to obtain a set of 3 numbers totalling 15 is the winner.

   TRUE or FALSE? With best play by both sides, the first player (Arthur) can force a win.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for every rational number q there exists a positive integer N such that $f^n(q) = 0$ for all $n \geq N$, where f^n denotes the n-fold iteration of f.

    TRUE or FALSE? For every real number t
$$\lim_{n\to\infty} f^n(t) = 0\,.$$

11. **TRUE or FALSE?**
$$\sum_{k=0}^{\infty} \frac{8k^3 + 4k + 1}{(2k)!} < \sum_{k=0}^{\infty} \frac{(2k+1)^3 + 4k + 3}{(2k+1)!}$$

12. Consider a sequence of positive integers
$$a_0\,, a_1\,, a_2\,, \ldots$$
    with the property that a_n equals the number of positive divisors of a_{n-1}. (The number a_i has both 1 and a_i as divisors.) We set $a_0 = 2000!$.

    TRUE or FALSE? For some positive integer n the number a_n is a perfect square.

13. **TRUE or FALSE?** There exists a function $f : [-1, +1] \to \mathbb{R}$ with continuous second derivative such that
$$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right) \quad \text{converges and} \quad \sum_{n=1}^{\infty} \left|f\left(\frac{1}{n}\right)\right| \quad \text{diverges.}$$

THE OLYMPIAD CORNER

No. 207

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

I hope you spent the break working on problems and that I will soon be receiving your nice solutions!

We first give some Olympiad sets. We start with the Swedish Mathematical Competition, Final Round, November, 1996. My thanks go to Richard Nowakowski for collecting them when he was Canadian Team Leader at the IMO in Argentina.

SWEDISH MATHEMATICAL COMPETITION

Final Round

November 23, 1996 — Time: 5 hours

1. Through an arbitrary interior point of a triangle, lines parallel to the sides of the triangle are drawn dividing the triangle into six regions, three of which are triangles. Let the areas of these three triangles be T_1, T_2, and T_3 and let the area of the original triangle be T. Prove that

$$T = \left(\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}\right)^2 .$$

2. In the country of Postonia, one wants to have only two values of stamps. The two values should be integers greater than one and the difference between the two should be two. It should also be possible to combine, in a precise way, stamps for each letter, the postage of which is greater than or equal to the sum of the two values. What values can be chosen?

3. For all integers $n \geq 1$ the functions p_n are defined for $x \geq 1$ by

$$p_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right) .$$

Show that $p_n(x) \geq 1$ and that $p_{mn}(x) = p_m(p_n(x))$.

4. A pentagon $ABCDE$ is inscribed in a circle. The angles at A, B, C, D, E form an increasing sequence. Show that the angle at C is greater than $\pi/2$. Also prove that this lower bound is best possible.

5. Let $n \geq 1$. Prove that it is possible to select some of the integers $1, 2, 3, \ldots, 2^n$ so that for all integers $p = 0, 1, \ldots, n-1$, the sum of k^p over all selected integers $k \in \{1, 2, 3, \ldots, 2^n\}$ is the same as the sum of k^p over all non-selected integers $k \in \{1, 2, 3, \ldots, 2^n\}$.

6. Tiles of dimension 6×1 are used to construct a rectangle. Prove that one of the sides has a length divisible by 6.

Next we give the problems of the 48th Polish Mathematical Olympiad, Final Round, written April 4–5, 1997. My thanks again go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina, and to Marcin E. Kuczma, Warszawa, Poland for sending me the problems.

48th POLISH MATHEMATICAL OLYMPIAD
Final Round – April 4–5, 1997
First Day — Time: 5 hours

1. The positive integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ satisfy the conditions:

$$x_6 = 144 \quad \text{and} \quad x_{n+3} = x_{n+2}(x_{n+1} + x_n) \quad \text{for} \quad n = 1, 2, 3, 4\,.$$

Compute x_7.

2. Solve the following system of equations in real numbers x, y, z:

$$\begin{aligned} 3(x^2 + y^2 + z^2) &= 1\,, \\ x^2y^2 + y^2z^2 + z^2x^2 &= xyz(x + y + z)^3\,. \end{aligned}$$

3. In a triangular pyramid $ABCD$, the medians of the lateral faces ABD, ACD, BCD, drawn from vertex D, form equal angles with the corresponding edges AB, AC, BC. Prove that the area of each lateral face is less than the sum of the areas of the two other lateral faces.

Second Day — Time: 5 hours

4. The sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 0\,, \qquad a_n = a_{[n/2]} + (-1)^{n(n+1)/2} \quad \text{for} \quad n > 1\,.$$

For every integer $k \geq 0$ find the number of all n such that

$$2^k \leq n < 2^{k+1}\,, \qquad a_n = 0$$

($[n/2]$ denotes the greatest integer not exceeding $n/2$).

5. Given is a convex pentagon $ABCDE$ with

$$DC = DE \quad \text{and} \quad \angle DCB = \angle DEA = 90^\circ .$$

Let F be the point on AB such that $AF : BF = AE : BC$. Show that

$$\angle FCE = \angle ADE \quad \text{and} \quad \angle FEC = \angle BDC .$$

6. Consider n points ($n \geq 2$) on the circumference of a circle of radius 1. Let q be the number of segments having those points as endpoints and having length greater than $\sqrt{2}$. Prove that $3q \leq n^2$.

Next we give the problems of the 18th Brazilian Mathematical Olympiad. Thanks again go to Richard Nowakowski for collecting and forwarding them to me when he was Canadian Team Leader to the IMO in Argentina.

18th BRAZILIAN MATHEMATICAL OLYMPIAD

1. Show that the equation

$$x^2 + y^2 + z^2 = 3xyz$$

has infinitely many integer solutions with $x > 0$, $y > 0$ and $z > 0$.

2. Is there a set A of n points ($n \geq 3$) in the plane such that:

(i) A does not contain three collinear points; and

(ii) given any three points in A, the centre of the circle which contains these points also belongs to A?

3. Let $f(n)$, $n \in \mathbb{Z}^+$, be the smallest number of ones that can be used to represent n using ones and any number of the symbols $+$, $\times$, (,), (with their usual meaning). For instance,

$$80 = (1+1+1+1+1) \times (1+1+1+1) \times (1+1+1+1)$$

and, therefore, $f(80) \leq 13$. Show that

$$3\log_3 n \leq f(n) < 5\log_3 n ,$$

for all $n > 1$. (Note: 11, 111, 1111, etc. may not be used in the expressions; only 1.)

4. Let D be a point of the side $\overline{BC}$ of the acute-angled triangle ABC ($D \neq B$ and $D \neq C$), O_1 be the circumcentre of $\triangle ABD$, O_2 be the circumcentre of $\triangle ACD$ and O be the circumcentre of $\triangle AO_1O_2$. Determine the locus described by the point O when D runs through the side $\overline{BC}$ ($D \neq B$ and $D \neq C$).

5. A set of marriages is *unstable* if two persons who are not married to each other prefer each other to their spouses. For instance, if Alessandra and Daniel are married and if Julia and Robinson are married, but Daniel prefers Julia to Alessandra, and Julia prefers Daniel to Robinson, then the set of marriages Alessandra-Daniel and Julia-Robinson is unstable. If the set of marriages is not unstable, we call it *stable*.

Consider now a group of people consisting of n boys and n girls. Each boy makes his own list ordering the n girls according to his preferences and, in the same way, each girl lists the n boys according to her preference. Show that it is always possible to marry the n boys and the n girls obtaining a stable marriage set.

6. Consider the polynomial $T(x) = x^3 + 14x^2 - 2x + 1$. Show that there exists a natural number $n > 1$ such that 101 divides $T^{(n)}(x) - x$ for all integers x. (Note: $T^{(n)}(x) = \underbrace{T(T(\cdots(T(x))))}_{n \text{ times}}$.)

As a fourth set to keep your solution skills finely honed, we give the problems from the Selection Test for the Vietnamese Team 1997, written May 16–17, 1997. Thanks again go to Richard Nowakowski who collected them for us when he was Canadian Team Leader to the IMO in Argentina.

SELECTION TEST FOR THE VIETNAMESE TEAM 1997

May 16–17, 1997

First Day — Time: 4 hours

1. Let $ABCD$ be a tetrahedron with $BC = a$, $CA = b$, $AB = c$, $DA = a_1$, $DB = b_1$, $CD = c_1$.

Prove that there exists one and only one point P satisfying the conditions:

$$\begin{aligned} PA^2 + a_1^2 + b^2 + c^2 &= PB^2 + b_1^2 + c^2 + a^2 \\ &= PC^2 + c_1^2 + a^2 + b^2 \;=\; PD^2 + a_1^2 + b_1^2 + c_1^2, \end{aligned}$$

and that for this point P, we have $PA^2 + PB^2 + PC^2 + PD^2 \geq 4R^2$, where R is the radius of the circumscribed sphere of the tetrahedron $ABCD$. Find a necessary and sufficient condition on the lengths of the edges so that the preceding inequality becomes an equality.

2. In a country, there are 25 towns. Determine the least number k such that one can set up flight routes connecting these towns (in both directions) so that the following conditions are simultaneously satisfied:

(i) from each town there are exactly k direct flight routes to k other towns,

(ii) if there is no direct flight route connecting two towns, then there exists at least one town which has direct flight routes to these two towns.

3. Find the greatest real number α such that there exists an infinite sequence of whole numbers (a_n) $(n = 1, 2, 3, \dots)$ satisfying simultaneously the following conditions:

(i) $a_n > 1997^n$ for every $n \in \mathbb{N}^*$,

(ii) $a_n^\alpha \leq U_n$ for every $n \geq 2$, where U_n is the greatest common divisor of the set of numbers $\{a_i + a_j \mid i + j = n\}$.

Second Day — Time: 4 hours

4. Let $f : \mathbb{N} \to \mathbb{Z}$ be the function defined by:

$f(0) = 2$, $f(1) = 503$, $f(n+2) = 503f(n+1) - 1996f(n)$ for all $n \in \mathbb{N}$.

For every $k \in \mathbb{N}^*$, take k arbitrary integers $s_1, s_2, \dots, s_k$ such that $s_i \geq k$ for all $i = 1, 2, \dots, k$ and for every s_i $(i = 1, 2, \dots, k)$, take an arbitrary prime divisor $p(s_i)$ of $f(2^{s_i})$.

Prove that for positive integers $t \leq k$, we have:

$$\sum_{i=1}^{k} p(s_i) \mid 2^t \quad \text{if and only if} \quad k \mid 2^t .$$

5. Determine all pairs of positive real numbers a, b such that for every $n \in \mathbb{N}^*$ and for every real root x_n of the equation

$$4n^2x = \log_2(2n^2x + 1)$$

we have

$$a^{x_n} + b^{x_n} \geq 2 + 3x_n .$$

6. Let three positive integers n, k, p satisfying $k \geq 2$ and $k(p+1) \leq n$, be given.

Let n distinct points on a circle be given. One colours these n points blue and red (each point by a colour) so that there exist exactly k points coloured blue, and on each arc, the extremities of which are two consecutive (in clockwise direction) blue points, there exist at least p points coloured red.

What is the number of such colourings? (Two such colourings are distinct if there exists at least one point coloured with two different colours by these colourings).

Next we give solutions by our readers to problems given in the February 1999 number of the *Corner*. We start with solutions to problems of the Bundeswettbewerb Mathematik, Second Round 1995 [1999 : 4].

1. Starting in $(1, 1)$, a stone is moved according to the following rules:

(i) From any point (a, b), it can be moved to $(2a, b)$ or $(a, 2b)$.

(ii) From any point (a, b), if $a > b$ it can be moved to $(a - b, b)$, and if $a < b$ it can be moved to $(a, b - a)$.

Determine a necessary and sufficient relation between x and y so that the stone can reach (x, y) after some moves.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give the solution by Aassila.

Since $\gcd(1, 1) = 1$ and noting that $\gcd(p, q) = \gcd(p, q - p)$, we conclude that an odd common divisor can never be introduced. Hence $\gcd(x, y) = 2^r$ for some $r \in \mathbb{N}$ if (x, y) is reachable.

Assume now that $\gcd(x, y) = 2^r$ and let us prove that (x, y) is reachable. From the pairs (p, q) from which (x, y) can be reached, we choose the pair which minimizes $p + q$. In fact $p = q$; indeed if $p > q$ then (p, q) is reachable from $(\frac{p+q}{2}, q)$, a contradiction; similarly for the case in which $p < q$. Since $\gcd(p, q) = 2^r$ and neither p nor q is even, otherwise $p = q$ is not minimal for (p, q) from which (x, y) can be reached. We conclude that $p = q = 1$ and then (x, y) is reachable.

2. In a segment of unit length, a finite number of mutually disjoint subsegments are coloured such that no two points with distance 0.1 are both coloured. Prove that the total length of the coloured subsegments is not greater than 0.5.

Comment by Mohammed Aassila, Strasbourg, France. [*Ed: A solution was received from Pierre Bornsztein, Courdimanche, France.*]

This problem appeared as problem 6 of the Swedish Mathematical Competition 1986 and a solution appeared in [1992 : 296].

3. Every diagonal of a given pentagon is parallel to one side of the pentagon. Prove that the ratio of the lengths of a diagonal and its corresponding side is the same for each of the five pairs. Determine the value of this ratio.

Solutions by Mohammed Aassila, Strasbourg, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.

In the pentagon $ABCDE$, we assume that $AB \| CE$, $BC \| AD$, $CD \| BE$, $DE \| CA$, and $EA \| DB$. As shown in figure 1 on the next page, we label the intersections of diagonals.

Since $ATDE$ and $SCDE$ are both parallelograms we get:

$AT = ED = SC$, so that $AS + ST = ST + TC$. Thus we have $AS = TC$. Hence $CE : AB = CS : SA = AT : TC = AD : BC$.

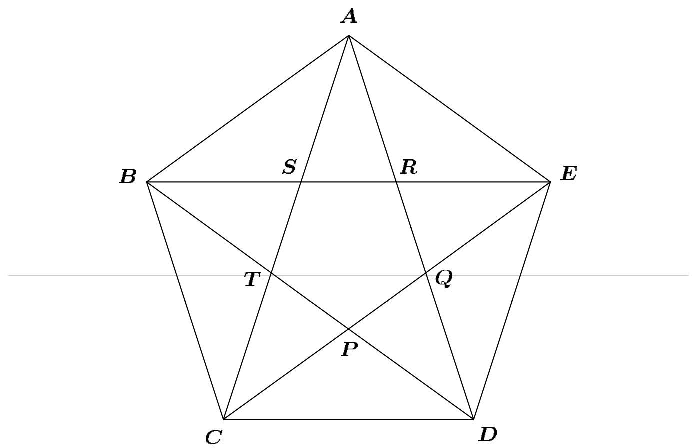


Figure 1.

Similarly we have

$$AD : BC = BE : CD = CA : DE = DB : EA .$$

We put

$$\frac{CE}{AB} = \frac{AD}{BC} = \frac{BE}{CD} = \frac{CA}{DE} = \frac{DB}{EA} = k .$$

Thus we have

$$\frac{CS}{SA} = \frac{CE}{AB} = k , \qquad \frac{DT}{BT} = \frac{AD}{BC} = k ,$$

and $\dfrac{DR}{RA} = \dfrac{BD}{AE} = k$, so that $\dfrac{DB}{BT} = \dfrac{k+1}{1}$ and $\dfrac{TS}{SA} = \dfrac{k-1}{1}$.

By Menelaus' Theorem for $\triangle ATD$ we get

$$\frac{DB}{BT} \cdot \frac{TS}{SA} \cdot \frac{AR}{RD} = 1 .$$

Therefore we have $\frac{k+1}{1} \cdot \frac{k-1}{1} \cdot \frac{1}{k} = 1$. Thus $k^2 - k - 1 = 0$, from which we obtain $k = \frac{1+\sqrt{5}}{2}$.

4. Prove that every integer k, $(k > 1)$ has a multiple which is less than k^4 and which can be written in decimal representation with at most four different digits.

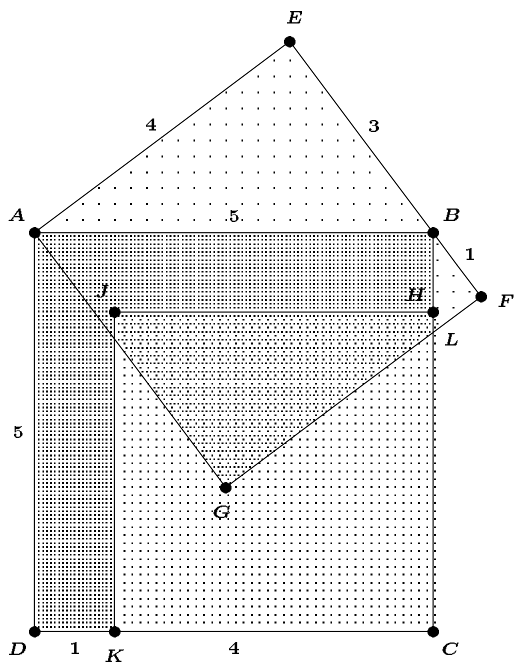
Comment by Mohammed Aassila, Strasbourg, France; and solution by Pierre Bornsztein, Courdimanche, France.

Aassila points out this problem was proposed by Poland but not used by the jury at the 31st IMO in China. A solution appeared in [1993 : 10].

Next we turn to the First Round 1996 of the Bundeswettbewerb Mathematik given in [1999 : 4].

1. Is it possible to cover a square of length 5 completely with three squares of length 4?

Solution by Sam Wong, student, Sir Winston Churchill High School, Calgary, Alberta.



This is a diagram showing the 5×5 square, $ABCD$, covered by two of the three available 4×4 squares. One 4×4 square shown is $JHCK$, and the other one is $AEFG$. The third 4×4 square is not shown, but occupies a similar position to square $AEFG$, except that it is reflected over a line joining AC. The reason that this square is not shown will be explained shortly.

The square $JHCK$ covers a 4×4 area of the square $ABCD$, and so leaves an L-shaped area (the polygon $ABHJKD$) left to be covered by the other two squares. To cover this L-shaped area with two squares, each square must cover, at the very least, half of this L-shaped area.

The square $AEFG$ does cover over half of the area. This can be proven as follows: Using Pythagoras' Theorem, the line EB can be calculated because it forms the triangle AEB and $AB = 5$, and $AE = 4$. Therefore, since $AB^2 = EB^2 + AE^2$, $EB^2 = 5^2 - 4^2$, or 9, so $EB = 3$.

On the left side of the diagram, $\angle BAG$ must be greater than $\angle BAC$. Since BAC is the diagonal of a square, $\angle BAC = 45^\circ$. To find $\angle BAG$, we must first find $\angle EAB$ (since $\angle EAB$ and $\angle BAG$ are complementary). The sine of $\angle EAB$ is $\frac{3}{5}$, so when the inverse sine is taken for $\angle EAB$, the angle is $36.870°$. Subtract this angle from $90°$, and we get $(90° - 36.870°)$ or $53.130°$. Thus, $\angle BAG$ is approximately $53°$, and $53° > 45°$. Thus $\angle BAG$ is greater than $\angle BAC$.

On the right side of the diagram, point L must extend past point H on the line BC, or BL must be greater than BH (and BH is 1 unit long). Since BL is the hypotenuse of $\triangle BFL$, and BF is 1, then BL must be greater than 1. Thus BL is greater than BH.

When the third unit four square is placed as the reflection of $AEFG$ over the line AC, then this square will cover a similar area to $AEFG$. Both squares together will completely cover the aforementioned L-shaped area, and together with square $JHCK$, the unit five square is completely covered with three unit four squares.

2. The cells of an $n \times n$–board are numbered according to the example shown for $n = 5$. You may choose n cells, not more than one from each row and each column, and add the numbers in the cells chosen. Which are the possible values of this sum?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Solutions by Pierre Bornsztein, Courdimanche, France; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Sam Wong, student, Sir Winston Churchill High School, Calgary, Alberta. We give Wang's solution.

Using the terminology of combinatorial matrix theory, a collection of n entries from an $n \times n$ matrix $A = (a_{ij})$ with no two entries lying in the same row or the same column, is called a *diagonal* and the sum of the entries from a diagonal is called a *diagonal sum*. Note that if σ is a permutation of $\{1, 2, \ldots, n\}$ then $\{a_{i\sigma(i)} \mid i = 1, 2, \ldots, n\}$ would be a diagonal of A, and conversely, every diagonal gives rise to a permutation. Thus, A has exactly $n!$ diagonals.

We show that for the matrix considered, *all* diagonal sums equal $n(n^2 + 1)/2$. To see this, note that by assumption, $a_{ij} = (i - 1)n + j$ for all $i, j = 1, 2, \ldots, n$. Hence

$$\begin{aligned}
\sum_{i=1}^{n} a_{i\sigma(i)} &= \sum_{i=1}^{n}((i-1)n+\sigma(i)) \\
&= \sum_{i=1}^{n}((i-1)n+i) \;=\; (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} n \\
&= \frac{n(n+1)^2}{2} - n^2 \;=\; \frac{n(n^2+1)}{2}\,.
\end{aligned}$$

Remark. The sum of all the $n!$ diagonal sums is

$$S = \sum_{\sigma}\sum_{i=1}^{n} a_{i\sigma(i)} = n!\,\frac{n(n^2+1)}{2}\,.$$

On the other hand, since each entry of A lies on exactly $(n-1)!$ diagonals,

$$\begin{aligned}
S \;=\; (n-1)!\sum_{i,j=1}^{n} a_{ij} &= (n-1)!\sum_{k=1}^{n^2} k \\
&= (n-1)!\,\frac{n^2(n^2+1)}{2} \;=\; n!\,\frac{n(n^2+1)}{2}\,,
\end{aligned}$$

and so we have a check!

3. There are four straight lines in the plane, each three of them determining a triangle. One of these straight lines is parallel to one of the medians of the triangle formed by the other three lines. Prove that each of the other straight lines has the same property.

Solutions by Michel Bataille, Rouen, France; Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We first give Seimiya's geometric solution.

Let the four lines be a, b, c, d, and let $\triangle ABC$ be determined by a, b, and c as shown in the figure. The line d intersects BC, CA, AB at P, Q, R respectively.

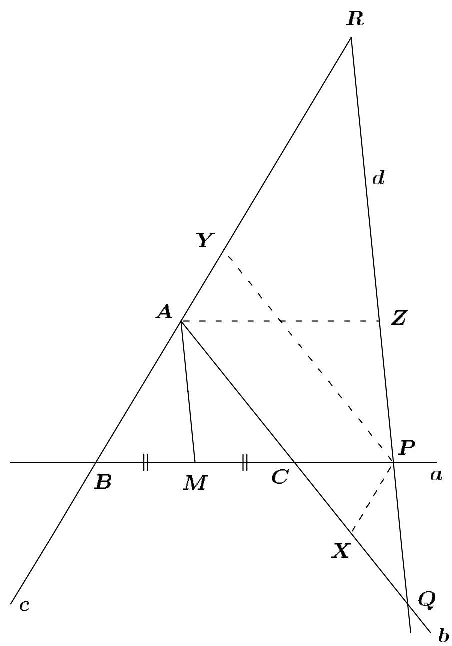
We assume that the median AM of $\triangle ABC$ is parallel to d. Let X be a point on AC such that $PX \| AB$. Since $PX \| AB$ and $AM \| PQ$, we have

$$\frac{AC}{CX} \;=\; \frac{BC}{CP} \;=\; \frac{2MC}{CP} \;=\; 2\cdot\frac{AC}{CQ}\,.$$

Thus $CQ = 2CX$; that is, $CX = XQ$.

Hence median PX of $\triangle PCQ$ is parallel to c. Let Y be a point on AB such that $PY \| AC$. Since $PY \| AC$ and $AM \| RP$, we have

$$\frac{BA}{BY} \;=\; \frac{BC}{BP} \;=\; \frac{2BM}{BP} \;=\; 2\cdot\frac{BA}{BR}\,.$$



Thus $BR = 2BY$; that is, $BY = YR$.

Hence median PY of $\triangle PBR$ is parallel to b. Let Z be a point on QR such that $AZ \| BP$. Since $AZ \| BP$ and $AM \| PQ$ we have

$$\frac{ZP}{ZQ} = \frac{AC}{AQ} = \frac{MC}{MP} = \frac{BM}{MP} = \frac{BA}{AR} = \frac{ZP}{ZR}\,.$$

Thus $ZQ = ZR$.

Hence median AZ of $\triangle AQR$ is parallel to a.

Next we give the solution of Bataille.

We denote the four straight lines by L_1, L_2, L_3, L_4, and we suppose that the names are chosen so that L_4 is parallel to one of the medians of the triangle formed by L_1, L_2, L_3. More precisely, let L_2, L_3 intersect at A, L_3, L_1 intersect at B, L_1, L_2 intersect at C and let us suppose $L_4 \| AM$ where M is the mid-point of BC. We shall work in the system of axes with origin A, and AB, AC as x–axis and y–axis respectively. Thus we have

$$A(0,0)\,; \quad B(1,0)\,; \quad C(0,1)\,; \quad M\left(\frac{1}{2},\frac{1}{2}\right)$$

and AM has equation $\quad y = x$.

We readily find:

$$L_1 : x + y \;=\; 1\,; \quad L_2 : x \;=\; 0\,; \quad L_3 : y \;=\; 0\,, \quad \text{and } L_4 : x - y \;=\; k\,,$$

where $k \neq 0,\, 1,\, -1$ [this condition on k ensures that L_4 determines a real triangle with any two of the three lines L_1, L_2, L_3].

It is now easy to compute the coordinates of the points A', B', C' where L_4 intersects L_1, L_2, L_3 respectively:

$$A'\left(\frac{1+k}{2}, \frac{1-k}{2}\right); \quad B'(0,-k); \quad C'(k,0)\,.$$

The mid-point I of BC' has coordinates $(\frac{1+k}{2}, 0)$ so that $\overrightarrow{A'I} = (0, \frac{k-1}{2})$. Hence the median $A'I$ (of the triangle formed by L_1, L_3, L_4) is parallel to L_2.

Similarly, the mid-point J of CB' has coordinates $(0, \frac{1-k}{2})$ and $\overrightarrow{A'J} = (-\frac{1+k}{2}, 0)$ is parallel to L_3.

Lastly, the mid-point K of $B'C'$ has coordinates $(\frac{k}{2}, -\frac{k}{2})$ and $\overrightarrow{AK} = (\frac{k}{2}, -\frac{k}{2})$ is parallel to L_1.

4. Determine the set of all positive integers n for which $n \cdot 2^{n-1}$ is a perfect square.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Let E be the set of such integers.

First Case. n is odd.

Write $n = 2k+1$ with $k \geq 0$. Then we have

$$n2^{n-1} \;=\; (2k+1)(2^k)^2 \quad \text{with } 2 \text{ and } 2k+1 \text{ coprime.}$$

Then $n \in E$ if and only if $2k+1 = n$ is a perfect square.

Second Case. n is even.

Write $n = 2^\alpha k$, $k \geq 1$ an odd integer and $\alpha \geq 1$. Then we have

$$n2^{n-1} \;=\; k2^{\alpha+2^\alpha k-1} \quad \text{with } k \text{ and } 2 \text{ coprime.}$$

Thus $n \in E$ if and only if $\alpha + 2^\alpha k - 1$ is even and k is a perfect square.

Thus $n \in E$ if and only if α is odd and k is a perfect square.

Further, $n \in E$ if and only if $n = 2b^2$ where $b \in \mathbb{N}^*$.

In summary, $n \in E$ if and only if n is an odd square or n is twice a square.

Now we turn to readers' comments and solutions to problems of the XLV Lithuanian Mathematical Olympiad 1996 given in [1999 : 5].

1. Solve the following equation in positive integers:

$$x^3 - y^3 \;=\; xy + 61\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Andrew Blinn, student, Western Canada High School, Calgary, Alberta; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Blinn's solution.

Manipulating the equation we obtain

$$(x-y)(x^2+xy+yz) - xy \;=\; 61$$

and

$$(x-y)((x-y)^2+3yx) - xy \;=\; 61\,.$$

Set $x - y = a$. Since the right-hand side is positive, $a = x - y > 0$. Set $b = xy$. Note that $b > 0$.

Rewriting the equation in terms of a and b yields

$$a^3 + b(3a-1) \;=\; 61\,,$$

where $a > 0$ and $b > 0$.

Thus $a^3 < 61$, so $a = 1, 2, 3$.

Also $b = \frac{61-a^3}{3a-1} \in \mathbb{Z}$, so trying

$$\begin{array}{rclcrcl} a &=& 1 & \text{gives} & b &=& 30\,, \\ a &=& 2 & \text{gives} & b &=& 53/5\,, \\ a &=& 3 & \text{gives} & b &=& 17/4\,. \end{array}$$

Thus $a = 1$, $x - y = 1$ and $xy = 30$, giving $(y+1)y = 30$ with $y = 5$ (rejecting $y = -6$) and $x = 6$.

The unique solution is $x = 6$, $y = 5$.

Comment. Aassila also points out that this problem was proposed at the 15$^{\text{th}}$ All-Union Mathematical Olympiad held in Alma Ata, and a solution in Russian appears in N.B. Vassiliev and A.A. Egorov, "The Problems of the All-Union Mathematical Competitions", Moscow: Nauka, 1988.

2. Sequences $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ are such that $a_1 > 0$, $b_1 > 0$, and

$$a_{n+1} = a_n + \frac{1}{b_n}\,, \quad b_{n+1} = b_n + \frac{1}{a_n}\,, \quad n \in \mathbb{N}\,.$$

Prove that

$$a_{25} + b_{25} \;>\; 10\sqrt{2}\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Heinz-Jürgen Seiffert, Berlin, Germany. We give the solution by Klamkin.

It follows that $a_2 + b_2 \geq 4$ and

$$a_{n+1} + b_{n+1} \;=\; (a_n + b_n)(1 + 1/a_n b_n) \;\geq\; (a_n + b_n) + 4/(a_n + b_n)\,.$$

Let $x_{n+1} = x_n + 4/x_n$ with $x_2 = 4$. Then since $x + 4/x$ is increasing for $x \geq 2$, we have $a_{n+1} + b_{n+1} \geq x_{n+1}$. Since $(x_{n+1})^2 = (x_n)^2 + (4/x_n)^2 + 8$,

$$(x_{n+1})^2 \;\leq\; (x_n)^2 + 8\,.$$

Summing the latter set of inequalities for $n = 2$ to $n - 1$, we obtain

$$(x_n)^2 \;\leq\; x_2^2 + 8(n-2) \;=\; 8n\,.$$

It then follows that

$$(x_{n+1})^2 \;\geq\; (x_n)^2 + 8 + 1/2n\,,$$

and summing this inequality for $n = 2$ to 24, we obtain

$$(x_{25})^2 \;\geq\; (x_2)^2 + 8(23) + (1/4 + 1/6 + \cdots + 1/48)\,.$$

Hence, $(x_{25})^2 > 16 + 184$ or $10\sqrt{2} < x_{25} \leq a_{25} + b_{25}$.

Next we give the generalization by Seiffert.

More generally: Let $u > 0$, $v > 0$, and $w > 0$. If the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ satisfy $a_1 > 0$, $b_1 > 0$, and

$$a_{n+1} \;=\; ua_n + \frac{v}{b_n}\,, \quad b_{n+1} \;=\; \frac{b_n}{u} + \frac{w}{a_n}\,, \quad n \in N\,,$$

then

$$a_n b_n \;>\; (n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}\,, \quad n \geq 3\,, \tag{1}$$

and

$$a_n + b_n \;>\; 2\sqrt{(n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}}\,, \quad n \geq 3\,. \tag{2}$$

First, we note that $a_n > 0$ and $b_n > 0$, $n \in \mathbb{N}$. We have

$$\begin{aligned} a_{k+1}b_{k+1} \;&=\; \left(ua_k + \frac{v}{b_k}\right)\left(\frac{b_k}{u} + \frac{w}{a_k}\right) \\ &=\; \frac{v}{u} + uw + a_k b_k + \frac{vw}{a_k b_k}\,, \quad k \in \mathbb{N}\,. \end{aligned}$$

Summing as k ranges from 1 to $n-1$, where $n \geq 3$, gives

$$\begin{aligned} a_n b_n &= (n-1)\left(\frac{v}{u} + uw\right) + a_1 b_1 + \sum_{k=1}^{n-1} \frac{vw}{a_k b_k} \\ &> (n-1)\left(\frac{v}{u} + uw\right) + a_1 b_1 + \frac{vw}{a_1 b_1} \\ &\geq (n-1)\left(\frac{v}{u} + uw\right) + 2\sqrt{vw}\,, \end{aligned}$$

where we have used the AM-GM-Inequality. This proves (1). Then (2) follows from (1) and $a_n + b_n \geq 2\sqrt{a_n b_n}$.

In the particular case $u = v = w = 1$, (1) and (2) give

$$a_n b_n \;>\; 2n \quad \text{and} \quad a_n + b_n \;>\; 2\sqrt{2n}\,, \quad n \geq 3\,.$$

With $n = 25$, we then have $a_{25} b_{25} > 50$ and $a_{25} + b_{25} > 10\sqrt{2}$.

Remark. If $u = 1$ and $v = w$, then $a_1 b_n = b_1 a_n$ for all $n \in \mathbb{N}$, as is easily verified by induction on n.

3. Two pupils are playing the following game. In the system

$$\begin{cases} *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \end{cases}$$

they alternately replace the asterisks by any numbers. The first player wins if the final system has a non-zero solution. Can the first player always win?

Solution by Pierre Bornsztein, Courdimanche, France.

Yes. Denote the system as follows:

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= 0\,, & (1) \\ b_1 x + b_2 y + b_3 z &= 0\,, & (2) \\ c_1 x + c_2 y + c_3 z &= 0\,. & (3) \end{aligned}$$

The first player chooses any number for b_2.

Then form the pairs (a_1, c_1), (a_2, c_2), (a_3, c_3), (b_1, b_3).

Each time the second player chooses a number from one pair, then the first player gives the same number to the other member of the pair. Thus at the end $a_1 = c_1$, $a_2 = c_2$, $a_3 = c_3$, $b_1 = b_3$. So (1) and (3) are equivalent.

And, since the system is homogeneous, it is consistent and must have infinitely many solutions, in particular a non-$(0, 0, 0)$ solution, and the first player wins.

4. How many sides has the polygon inscribed in a given circle and such that the sum of the squares of its sides is the largest one?

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's solution.

We will prove that the "maximal" polygon is the equilateral triangle.

Lemma. Let $\mathcal{P} = A_1A_2\ldots A_n$ be a convex polygon with $n \geq 5$ sides. Then $\mathcal{P}$ has an obtuse angle.

Proof of the lemma. We have $\sum_{i=1}^{n} \widehat{A_i} = (n-2)\pi$. Suppose, on the contrary, that each $\widehat{A_i} \leq \frac{\pi}{2}$. Then, $\sum_{i}^{n} A_i \leq \dfrac{n\pi}{2}$. Thus $(n-2)\pi \;\leq\; \dfrac{n\pi}{2}$. Then $n \leq 4$, which is a contradiction.

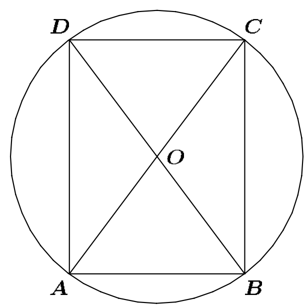
Let $\mathcal{P}$ be a convex polygon with $n \geq 4$ sides and with an obtuse angle. Let $\widehat{B} > \frac{\pi}{2}$ and A, C be the neighbouring vertices of B. Then, from the Law of Cosines:

$$AC^2 \;=\; AB^2 + BC^2 - 2AB \cdot BC\cos\widehat{B} \;>\; AB^2 + BC^2\,.$$

Thus, $\mathcal{P}$ is not maximal, because we have a larger sum by deleting B. Then, using the above lemma: if a "maximal" polygon exists, it has $n \leq 4$ sides, and no obtuse angle.

Let $\mathcal{P}_n$ be a convex polygon with $n \leq 4$ sides, no obtuse angle, inscribed in the circle $\mathcal{C}$ with centre O and radius R.

Let $n = 4$. The proof of the lemma may be used to prove that all angles of $\mathcal{P}_n$ are $\frac{\pi}{2}$ (because they are $\leq \frac{\pi}{2}$). Then $\mathcal{P}_4$ is a rectangle.



Pythagoras' Theorem leads to:

$$AB^2 + BC^2 \;=\; 4R^2 \;=\; CD^2 + DA^2\,.$$

Thus the sum of the squares of the sides is $S_4 \;=\; 8R^2$. Note that S_4 is independent of the rectangle.

Let $n = 3$. Suppose that $\triangle ABC$ is a non-obtuse triangle inscribed in $\mathcal{C}$. Let G be the centre of gravity of $\triangle ABC$. For any point M, we have

$$AM^2 + BM^2 + CM^2 \;=\; AG^2 + BG^2 + CG^2 + 3GM^2\,. \qquad (1)$$

Then

$$AG^2 + BG^2 + CG^2 + 3GO^2 = 3R^2 . \qquad (2)$$

Moreover, for $M = A$, we have $AB^2 + AC^2 = 4AG^2 + BG^2 + CG^2$. We also have the same relation for $M = B$, $M = C$.

Then, summing these three relations, we obtain

$$2(AB^2 + BC^2 + CA^2) = 6(AG^2 + BG^2 + CG^2)$$

and, using (2) we get

$$AB^2 + BC^2 + CA^2 = 9(R^2 - OG^2) .$$

We deduce that the sum of the squares is $S_2 \leq 9R^2$ with equality if and only if $O = G$; that is, $\triangle ABC$ is equilateral.

Then, the "maximal" polygon exists; it is the equilateral triangle and the sum is $S = 9R^2$.

5. Given ten numbers 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, one must cross out several of them so that the total of any of the remaining numbers would not be an exact square (that is, the sum of any two, three, four, ..., and of all the remaining numbers would not be an exact square). At most how many numbers can remain?

Solutions by Mohammed Aassila, Strasbourg, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The maximum number is five. First note that the numbers in each of the following sets add up to a perfect square: $\{2, 7\}$, $\{3, 6\}$, $\{3, 13\}$, $\{5, 11\}$, $\{6, 10\}$, $\{12, 13\}$, $\{3, 10, 12\}$. (And there are many more such sets, of course.) Let $S = \{2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13\}$ and partition S as $S = \{8\} \cup \{2, 7\} \cup \{5, 11\} \cup \{3,\ 6,\ 10,\ 12,\ 13\}$. Let $T \subseteq S$ be a set whose elements never add up to a perfect square and suppose $|T| \geq 6$. Then by the strong Pigeonhole Principle we have $|T \cap \{3,\ 6,\ 10,\ 12,\ 13\}| \geq 3$ which is impossible since if $3 \notin T$, then either $\{6, 10\} \subseteq T$ or $\{12, 13\} \subseteq T$, a contradiction, and if $3 \in T$, then $6 \notin T$ and $13 \notin T$ would imply that $\{3,\ 10,\ 12\} \subseteq T$, again a contradiction. Therefore, $|T| \leq 5$. On the other hand, 5-element subsets of S whose elements do not add up to perfect squares do exist. One such set is $T = \{3,\ 7,\ 8,\ 11,\ 12\}$ since direct checkings show that the values of the sums of any k elements of T, $(1 \leq k \leq 5)$ are: 3, 7, 8, 10, 11, 12, 14, 15, 18, 19, 20, 21, 22, 23, 26, 27, 29, 30, 31, 33, 34, 38, and 41.

That completes the *Corner* for this issue. Send me your Olympiad Contest materials and your nice solutions to problems from the *Corner*.

BOOK REVIEWS

ALAN LAW

Geometry from Africa — Mathematical and Educational Explorations
by Paulus Gerdes,
published by the Mathematical Association of America, 1999,
ISBN 0-88385-715-4, softcover, 244+ pages, $39.95 (U.S.)
Reviewed by **Julia Johnson**, *University of Regina, Regina, Saskatchewan.*

The thesis of this book is that the African peoples are actually doing mathematics in their art. The author has a prestigious and prolific teaching and research career. This book is a continuation of his inquiry into an emerging field known as *ethnomathematics*. Particular formulations of mathematical ideas develop from cultural activities. The aim of ethnomathematics is to uncover the common cognitive concepts underlying different cultures.

The book is divided into four parts which can be read independently from each other. The first part is entitled "*On geometrical ideas in Africa South of the Sahara*". The peoples of this area have been particularly active in geometrical thinking expressed in diverse cultural and social artifacts that exhibit a high degree of symmetry. Examples include rock paintings found in Northern Mozambique, petroglyphs from extreme East Angola, Adinkra stamp motifs, archaeological finds of carved or wooden patterns from the Republic of Mali and the geometrical structure of the Great Hall of King Munza. This part of the book is filled with photographs and sketches to illustrate the geometrical accents of African culture. The illustrations are annotated with their area of origin and sometimes time period (e.g., design from royal cloth from Northwest Cameroon, design from Nigeria embroidery, body painting on wooden sculpture collected in Niangara in 1910). The sketch of a roof structure of a Fulani house in Cameroon is mesmerizing, reminiscent of today's test pattern for astigmatism. Beautifully illustrated decorative designs on drums, facial tattoos from South Africa, semi-spherical basket fish traps, and the hexagonally woven bottom of a conical quail basket leave the reader with a graphic realization of what Cameroon mathematician George Njock said: that "*Black art is mathematics*". The pictures reveal vividly the geometry evident in every sphere of African life.

It is interesting to note Africa's connection with Egypt in terms of their similarity in originating mathematical ideas. For example, the Pythagorean proposition can be proved from designs derived from Mozambican decoration. Part 2 of the book, like Part 1, is filled with graphics, but in Part 2 the graphics are used to illustrate the steps in the production of the proof of the Pythagorean theorem. The theorem is proved again in a different way, this time starting with Chokwe sand drawings with fourfold symmetry which are easily transformed into Pythagorean designs. Part 2 is concluded with coverage of other connections between African art and variations of the

Theorem of Pythagoras. For example, the theorem can also be discovered in mat weaving patterns.

Part 3 explores the uses of African art forms for teaching notions of symmetry. This part of the book provides a careful enumeration and organization of many examples to further illustrate the appearance of geometrical ideas in African artifacts. Alternative ways are developed for rectangle constructions and for the determination of areas of circles and volumes of spatial figures.

Part 4 provides a summary of the author's previous books *Sona Geometry* (first published 1993/94 with several later versions and translations), *Luscona: Geometrical Recreations of Africa* (published 1991, translated 1997), and *Lunda Geometry* (1995). As such, it provides an educational exploration of geometry focusing on the sona sand drawing tradition among the Chokwe people in Southern Central Africa. This part of the book, like the others, is well illustrated with visual representations.

The pictorials that appear abundantly throughout all four parts convey African art as mathematical symmetry. The educational uses of African art forms teach geometrical notions in a very effective way. A research methodology for advancing the field of ethomathematics is evident from this text. Paraphrasing Gerdes himself : If one tries to vary the geometrical forms and patterns of traditional objects, a sub-optimal solution is reached which disallows many practical advantages. This is to say that the traditional form is never arbitrary but embodies mathematical knowledge that expresses itself as accumulated knowledge and wisdom (1997; 1986).

I highly recommend this book both from the mathematical and artistic points of view.

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Some bounds for $\phi(n)\sigma(n)$

Edward T.H. Wang

Let $\mathbb{N}$ denote the set of all natural numbers. In elementary number theory, three multiplicative functions which are discussed most frequently are $\tau(n)$, the number of (positive) divisors of $n \in \mathbb{N}$; $\sigma(n)$, the sum of all the divisors of n; and $\phi(n)$, Euler's totient function; that is, the number of positive integers in $\{1, 2, \ldots, n\}$ which are coprime with n.

Though there are well-known formulae for computing the exact values of these functions, these formulae all depend on the actual prime power factorization of the natural number n. Hence it is of interest to find upper and lower bounds for these functions, preferably in terms of n only. Indeed, examples of such bounds abound in the literature. For example, the statements in the following proposition are clearly true and trivial.

Proposition 1. Let $n \in \mathbb{N}$, $n \geq 2$. Then

(a) $\tau(n) \geq 2$,

(b) $\phi(n) \leq n - 1$,

(c) $\sigma(n) \geq n + 1$.

In each of the three inequalities above, equality holds if and only if n is a prime.

On the other hand, there are less trivial bounds for these functions. For example, the inequalities in the next proposition can be found in ([1], p. 214 and p. 222).

Proposition 2. Let $n \in \mathbb{N}$. Then

(a) n is composite if and only if $\phi(n) < n - \sqrt{n}$,

(b) n is composite if and only if $\sigma(n) > n + \sqrt{n}$.

In view of Proposition 1, (b) and (c), it is natural to ask whether there is any inequality between $\phi(n)\sigma(n)$ and $n^2 - 1$. A quick numerical checking for $n \geq 2$ seems to suggest that $\phi(n)\sigma(n) \leq n^2 - 1$. In our first result (Proposition 4 below), we will show that this is indeed the case, but first, we need a lemma.

Lemma 3. Let $a_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$.
Then $(a_1 - 1)(a_2 - 1)\cdots(a_k - 1) \leq a_1 a_2 \cdots a_k - 1$. Equality holds if and only if either $k = 1$ or $a_i = 1$ for all i, $1 \leq i \leq k$.

Proof. We clearly have equality when $k = 1$. Hence we assume that $k \geq 2$. Let $a_i = 1 + b_i$ where $b_i \geq 0$ for all $i = 1, 2, \ldots, k$. Then

$$\left(\prod_{i=1}^{k} a_i\right) - 1 - \prod_{i=1}^{k}(a_i - 1) \;=\; \left(\prod_{i=1}^{k}(1 + b_i)\right) - 1 - \prod_{i=1}^{k} b_i \;=\; \sum_{m=1}^{k-1} S_m \;\geq\; 0\,,$$

where S_m is the m^{th} elementary symmetric function defined to be the sum of all $\binom{k}{m}$ possible products of the b_i's taken m at a time.

If equality holds, then $S_m = 0$ for all $m = 1, 2, \ldots, k-1$, from which it follows that $b_i = 0$ or $a_i = 1$ for all $i = 1, 2, \ldots, k$. This completes the proof. ∎

We now recall the familiar formulae for $\phi(n)$ and $\sigma(n)$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime power factorization of $n \in \mathbb{N}$, then

(a) $\phi(n) = n \displaystyle\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$,

(b) $\sigma(n) = \displaystyle\prod_{i=1}^{k}\left(1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i}\right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$.

Now we are ready for our first result.

Proposition 4. Let $n \in \mathbb{N}$, $n \geq 2$. Then $\phi(n)\sigma(n) \leq n^2 - 1$. Equality holds if and only if n is a prime.

Proof. First suppose n has more than one prime factor. Thus, let $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$ be the prime power factorization of n where $k \geq 2$. Since ϕ and σ are multiplicative functions, we have by Lemma 3, that

$$\begin{aligned} \phi(n)\sigma(n) \;&=\; \prod_{i=1}^{k}(\phi(p_i^{\alpha_i})\sigma(p_i^{\alpha_i})) \;\leq\; \prod_{i=1}^{k}\left(p_i^{2\alpha_i} - 1\right) \\ &<\; \left(\prod_{i=1}^{k} p_i^{2\alpha_i}\right) - 1 \;=\; n^2 - 1\,. \end{aligned}$$

Next suppose $n = p^\alpha$ for some prime p where $\alpha \in \mathbb{N}$. Then we have

$$\begin{aligned} \phi(n)\sigma(n) \;&=\; p^{\alpha-1}(p-1)\,\frac{p^{\alpha+1} - 1}{p - 1} \;=\; p^{\alpha-1}\left(p^{\alpha+1} - 1\right) \\ &=\; p^{2\alpha} - p^{\alpha-1} \;\leq\; p^{2\alpha} - 1 \;=\; n^2 - 1 \end{aligned}$$

with equality if and only if $\alpha = 1$; that is, if and only if n is a prime. This completes the proof. ∎

Similarly, the two bounds in Proposition 2 raise the following natural question: Is there any inequality between $\phi(n)\sigma(n)$ and $n^2 - n$? Searching for a possible answer to this question we first do some computations and compare the values of $\phi(n)\sigma(n)$ with $n^2 - n$ for all n, $2 \le n \le 49$, as shown in the table below. Note that those integers n for which $\phi(n)\sigma(n) > n^2 - n$ are circled.

n	$\phi(n)\sigma(n)$	$n^2 - n$	n	$\phi(n)\sigma(n)$	$n^2 - n$	n	$\phi(n)\sigma(n)$	$n^2 - n$
2	3	2	18	234	306	34	864	1122
3	8	6	19	360	342	35	1152	1190
4	14	12	20	336	380	36	1092	1260
5	24	20	21	384	420	37	1368	1332
6	24	30	22	360	462	38	1080	1406
7	48	42	23	528	506	39	1344	1482
8	60	56	24	480	552	40	1440	1560
9	78	72	25	620	600	41	1680	1640
10	72	90	26	504	650	42	1152	1722
11	120	110	27	720	702	43	1848	1806
12	112	132	28	672	756	44	1680	1892
13	168	156	29	840	812	45	1872	1980
14	144	182	30	576	870	46	1584	2070
15	192	240	31	960	930	47	2208	2162
16	248	240	32	1008	992	48	1984	2256
17	288	272	33	960	1056	49	2394	2352

The above table seems to indicate that $\phi(n)\sigma(n)$ is greater than $n^2 - n$ ***exactly*** when n is a prime power. Our second result below shows that this is indeed the case.

Proposition 5. Let $n \in \mathbb{N}$, $n \ge 2$. Then $\phi(n)\sigma(n) > n^2 - n$ if and only if $n = p^k$ for some prime p where $k \in \mathbb{N}$.

Proof. The sufficiency is clear since if $n = p^k$, then

$$\phi(n)\sigma(n) \;=\; p^{2k} - p^{k-1} \;>\; p^{2k} - p^k \;=\; n^2 - n\,.$$

To prove the necessity, we show that if $n \ne p^k$, then in fact, $\phi(n)\sigma(n) < n^2 - n$. Assume first that n has only two distinct prime factors, so $n = p^\alpha q^\beta$ where α, $\beta \in \mathbb{N}$ and p and q are distinct primes. Then using the multiplicative property of ϕ and σ, we have

$$\phi(n)\sigma(n) \;=\; \phi(p^\alpha)\sigma(p^\alpha)\phi(q^\beta)\sigma(q^\beta) \;=\; \left(p^{2\alpha} - p^{\alpha-1}\right)\left(q^{2\beta} - q^{\beta-1}\right)$$

and so

$$\begin{aligned} n^2 - n - \phi(n)\sigma(n) &= p^{2\alpha}q^{2\beta} - p^\alpha q^\beta - \left(p^{2\alpha} - p^{\alpha-1}\right)\left(q^{2\beta} - q^{\beta-1}\right) \\ &= p^{2\alpha}q^{\beta-1} + p^{\alpha-1}q^{2\beta} - p^\alpha q^\beta - p^{\alpha-1}q^{\beta-1} \\ &= p^{\alpha-1}q^{\beta-1}(p^{\alpha+1} + q^{\beta+1} - pq - 1) \end{aligned}$$

$$\begin{aligned} &\geq p^{\alpha-1}q^{\beta-1}(p^2+q^2-pq-1) \\ &= p^{\alpha-1}q^{\beta-1}((p-q)^2+pq-1) \\ &> 0\,. \end{aligned}$$

Therefore, $\phi(n)\sigma(n) < n^2 - n$.

Now suppose $\phi(n)\sigma(n) < n^2 - n$ holds for all $n \in \mathbb{N}$ with t distinct prime factors for all $t = 2, 3, \ldots, m$ for some $m \geq 2$, and suppose $n \in \mathbb{N}$ has $m+1$ distinct prime factors. Then we can write $n = p^k s$ where p is a prime and $s \in \mathbb{N}$ has m distinct prime factors and $(p, s) = 1$. Hence

$$\begin{aligned} \phi(n)\,\sigma(n) &= \phi(p^k)\sigma(p^k)\phi(s)\sigma(s) \\ &= \left(p^{2k} - p^{k-1}\right)\phi(s)\sigma(s) \\ &< \left(p^{2k} - p^{k-1}\right)(s^2 - s)\,, \end{aligned}$$

where the inequality is by the induction hypothesis. Therefore

$$\begin{aligned} n^2 - n - \phi(n)\sigma(n) &> p^{2k}s^2 - p^k s - (p^{2k} - p^{k-1})(s^2 - s) \\ &= p^{2k}s + p^{k-1}s^2 - p^k s - p^{k-1}s \\ &= p^{k-1}s(p^{k+1} - p + s - 1) \\ &> 0\,, \end{aligned}$$

from which $\phi(n)\sigma(n) < n^2 - n$ follows and our induction is complete. ∎

The corollary below clearly follows from Propositions 4 and 5:

Corollary 6. $\displaystyle \limsup_{n\to\infty} \frac{\phi(n)\sigma(n)}{n^2} = 1$.

Acknowledgement: The author would like to thank the referees for their careful reading of the original manuscript and for making a number of valuable observations and suggestions which greatly improve the clarity of this paper.

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Letter to the Editor

Regarding: Trevor Lipscombe & Arturo Sangalli; The Devil's dartboard [2000 : 215].

The arrangement of the numbers on a dartboard has intrigued the many players of the game and a number of mathematicians. I have outlined the history of the dartboard in my paper: Arranging a dartboard [*Bull. Inst. Math. Appl.* 16:4 (Apr 1980) 93-97; *CMP* 12:22 (1980) 1446; *MR* 81j:05005]. A simpler approach than used by Lipscombe & Sangalli is based on the idea that large and small numbers tend to alternate, so for a cyclic arrangement (a_i) of the first n integers, one could consider the sums of two adjacent numbers, $s_i = a_i + a_{i+1}$, and seek to make these as equal as possible; that is, have the least standard deviation or variance. This approach is much easier. It was used by Selkirk to determine the best and worst distributions, but the construction was a bit vague. I had also used the same approach and found an easy way to determine these optimal distributions. First one notes that minimizing the variance corresponds to minimizing the sum $C = \sum a_i a_{i+1}$. I considered a section $\ldots, a, b, \ldots, c, d, \ldots$ of the distribution and noted that reversing the portion from b to c reduced (or preserved) C unless $(a - d)(c - b) > 0$. But there is essentially only one distribution which satisfies this necessary condition and it looks like: $\ldots, 4, n-2, 2, n, 1, n-1, 3, n-3, 5, \ldots$ I extended the analysis, relating it to the auto-correlation coefficient of the cycle with itself shifted by one and determining the mean and standard deviation of this auto-correlation, from which one can reasonably deduce that the designer of the standard dartboard must have had something like the idea of putting big numbers next to little ones in his mind.

I also considered making other sums as equal as possible, of which the most natural next stage is $s_i = a_{i-1} + a_i + a_{i+1}$, as considered by Lipscombe & Sangalli, and, more generally, $s_i = \sum p_j a_{i+j}$, where p_d is the probability of hitting the value which is d values from the one aimed at. Then the average of the s_i is the same as the average of the a_i, which is $\bar{a} = (n+1)/2$. Setting $D_d = \sum (a_i - \bar{a})(a_{i+d} - \bar{a})$, we have that D_d/nv is the auto-correlation coefficient of the cycle with itself shifted by d places — here v is the variance of the first n integers, namely $(n^2 - 1)/12$. Straightforward manipulation gives us an expression for the variance V of the s_i as $nV = \sum\limits_{j,k} D_{k-j} p_j p_k$.
We have $D_0 = nv$; $D_d = D_{-d} = D_{n-d}$ and $\sum D_i = 0$, which can be used to simplify the expression for nV. One usually also assumes symmetry of the p_d, but even so, the problem generally involves at least two D_d and different choices of the p_d will give different optima and a given set of probabilities may have several optima.

For the version considered by Lipscombe & Sangalli, we take $p_{-1} = p_0 = p_1 = 1/3$ and all other probabilities equal to 0, so we have $9nV = 3D_0 + 4D_1 + 2D_2$. For $n = 6$, the unique best distribution is $1, 6, 3, 2, 5, 4$, as also found by Lipscombe & Sangalli. However, for the simpler version considered above, corresponding to $p_0 = p_1 = 1/2$ and $4nV = 2D_0 + 2D_1$, the unique best distribution is $1, 6, 2, 4, 3, 5$. Returning to $p_{-1} = p_0 = p_1 = 1/3$, the case $n = 7$ has three best distributions: $1, 4, 7, 2, 3, 5, 6$; $1, 4, 7, 3, 2, 5, 6$; $1, 4, 7, 3, 2, 6, 5$. These are rather better than the distribution given by Lipscombe & Sangalli's algorithm, which I find is $7, 1, 4, 6, 2, 5, 3$. The technique of reversing a part of the distribution can be used here, but it leads to messy conditions which do not necessarily force a global minimum, though a computer could easily use them to improve an approximate minimum. The simplest case is reversing two adjacent terms in the arrangement; changing $\ldots, a, b, c, d, e, f, \ldots$ to $\ldots, a, b, d, c, e, f, \ldots$ decreases (preserves) the variance if $(a + b - e - f)(c - d) > 0 (= 0)$. For the result of Lipscombe & Sangalli, no such exchange reduces the variance, but exchanging 2 and 5 preserves the variance and in that arrangement, exchanging 1 and 4 does reduce the variance and gives a minimal arrangement.

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Ian Cook. Unbiased dartboards and biased calculators. *Math. Gaz.* 61 (No.417) (1977) 187-191. [He corrects and extends Selkirk, but considers different measures than treated here.]

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THE SKOLIAD CORNER

No. 47

R.E. Woodrow

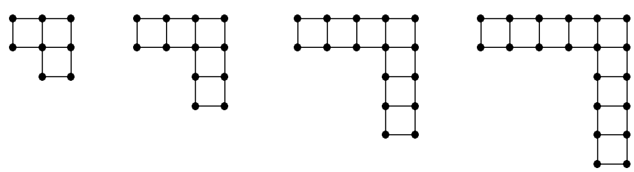
This issue we give the preliminary round of the Junior High School Mathematics Contest of the British Columbia Colleges. This was written in the schools by Grade 8 to 10 students on March 8, 2000. Thanks go to Jim Totten, The University College of the Cariboo, one of the organizers, for forwarding the contest materials to us.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. After 15 litres of gasoline was added to a partially filled fuel tank, the tank was 75% full. If the tank's capacity is 28 litres, then the number of litres in the tank before adding the gas was:

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

2. The following figures are made from matchsticks.



If you had 500 matchsticks, the number of squares in the largest such figure you could build would be:

(a) 164 (b) 165 (c) 166 (d) 167 (e) none of these

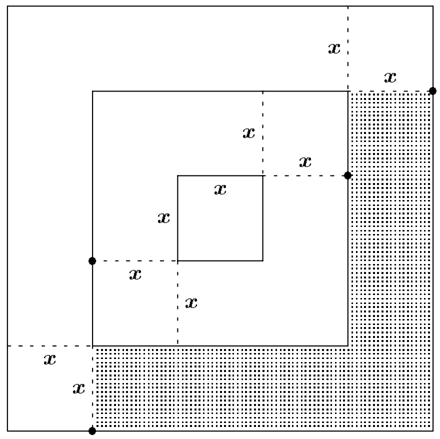
3. The perimeter of a rectangle is 56 metres. The ratio of its length to width is $4 : 3$. The length, in metres, of a diagonal of the rectangle is:

(a) 17.5 (b) 20 (c) 25 (d) 40 (e) none of these

4. If April 23 falls on Tuesday, then March 23 of the same year was a:

(a) Saturday (b) Sunday (c) Monday (d) Wednesday (e) Thursday

5. Consider the dart board shown in the diagram. If a dart may hit any point on the board with equal probability, the probability it will land in the shaded area is:

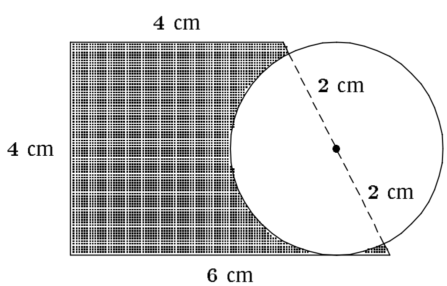


(a) 0.07 (b) 0.24 (c) 0.25 (d) 0.28 (e) 0.32

6. The proper divisors of a number are those numbers that are factors of the number other than the number itself. For example, the proper divisors of 12 are 1, 2, 3, 4 and 6. An *abundant* number is defined as a number for which the sum of its proper divisors is greater than the number itself. For example, 12 is an abundant number since $1 + 2 + 3 + 4 + 6 > 12$. Another example of an abundant number is:

(a) 13 (b) 16 (c) 30 (d) 44 (e) 50

7. The figure below is a right trapezoid with side lengths 4 cm, 4 cm, and 6 cm as labelled. The circle has radius 2 cm. The area, in cm^2, of the shaded region is:

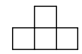


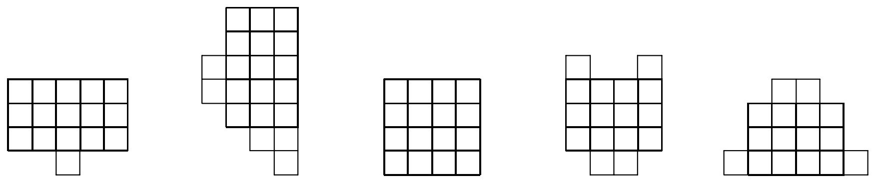
(a) $20 - 4\pi$ (b) 16 (c) $24 - 2\pi$ (d) $20 - 2\pi$ (e) $16 + 2\pi$

8. Three vertices of parallelogram $PQRS$ were $P(-3, -2)$, $Q(1, -5)$, and $R(9, 1)$ with P and R diagonally opposite. The sum of the coordinates of vertex S is:

(a) 13 (b) 12 (c) 11 (d) 10 (e) 9

9. Which shape *cannot* be filled, without any overlapping, using copies of the tile shown on the right?





(a) (b) (c) (d) (e)

10. Arrange the following in ascending order:

$$2^{5555} \qquad 3^{3333} \qquad 6^{2222}$$

(a) 2^{5555} 3^{3333} 6^{2222}
(b) 2^{5555} 6^{2222} 3^{3333}
(c) 6^{2222} 3^{3333} 2^{5555}
(d) 3^{3333} 6^{2222} 2^{5555}
(e) 3^{3333} 2^{5555} 6^{2222}

11. 2000 days, 2000 hours, 2000 minutes, and 2000 seconds would be equivalent to N million seconds. Of the choices offered, the closest approximation of N is:

(a) 1 (b) 15 (c) 45 (d) 180 (e) 2000

12. A three-digit decimal number abc may be expressed as $100a + 10b + c$ where each of the digits is multiplied by its respective place value and subsequently summed. If $a = b = c$ and $a > 0$, which of the following numbers must be a factor of the three digit number abc?

(a) 7 (b) 11 (c) 13 (d) 19 (e) 37

13. If $(x + y)^2 - (x - y)^2 > 0$, then

(a) ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$)
(b) $x > 0$ and $y < 0$
(c) $x < 0$ and $y > 0$
(d) ($x > 0$ and $y < 0$) or ($x < 0$ and $y > 0$)
(e) $x > y$ or $x < y$

14. Consider all non-congruent triangles with all sides having whole number lengths and a perimeter of 12 units. The following statements correspond to these triangles.

(i) There are only three such triangles.
(ii) The number of equilateral triangles equals the number of scalene triangles.
(iii) None of these triangles are right angled.
(iv) None of these triangles have a side of length 1 unit.

Of the four statements made, the number of true statements is:

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

15. An altitude, h, of a triangle is increased by a length m. How much must be taken from the corresponding base, b, so that the area of the new triangle is one-half that of the original?

(a) $\frac{bm}{h+m}$ (b) $\frac{bh}{2(h+m)}$ (c) $\frac{b(2m+h)}{m+h}$ (d) $\frac{b(m+h)}{2m+h}$ (e) $\frac{b(2m+h)}{2(h+m)}$

Last issue we gave the first round of two contests. Next we give short "official" solutions to the first of them. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Buenos Aires, for collecting them.

BUNDESWETTBEWERB MATHEMATIK
Federal Contest in Mathematics (Germany) 1997
First Round

1. Can you always choose 15 from 100 arbitrary integers so that the difference of any two of the chosen integers is divisible by 7?

What is the answer if 15 is replaced by 16? (Proof!)

Solution. The answer to the first question is *yes*. This is an elementary application of the Pigeon-Hole Principle, as one remainder $\pmod 7$ must occur at least 15 times.

The answer to the second question is *no*. For a contradiction, take the integers from 1 to 100.

2. Determine all primes p for which the system

$$\begin{aligned} p+1 &= 2x^2\,, \\ p^2+1 &= 2y^2\,, \end{aligned}$$

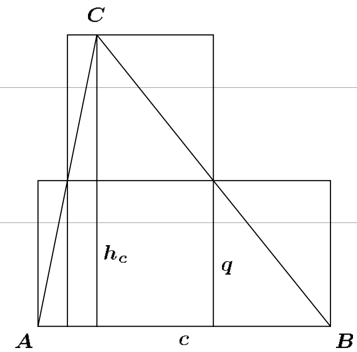
has a solution in integers x, y.

Solution. We can assume $y > x \geq 2$ and $p > y$. Subtracting the given equations, we have $p(p-1) = 2(y-x)(y+x)$. It follows $p > y - x$ and $2p > y + x$, yielding $p = x + y$ and $p - 1 = 2(y - x)$. Eliminating y from the given equations, we get $p + 1 = 4x \Longrightarrow x = 2 \Longrightarrow p = 7$. Indeed, 7 satisfies the conditions.

3. A square S_a is inscribed in an acute-angled triangle ABC by placing two corners on the side BC and one corner on AC and AB, respectively. In a similar way, squares S_b and S_c are inscribed in ABC.

For which kind of triangle ABC do the sides of S_a, S_b and S_c have equal length?

Solution.



The inscribed squares are congruent if and only if triangle ABC is equilateral. The if-direction is trivial, so let us assume the squares to be congruent. Denoting the area of ABC by A and the length of the squares by q, we have $A = q^2 + \frac{1}{2}q(h_c - q) + \frac{1}{2}(c - q)q = \frac{1}{2}q(h_c + c)$. Cyclic permutation leads to $A = \frac{1}{2}q(h_b + b) = \frac{1}{2}q(h_a + a)$, thus $h_a + a = h_b + b$. With $2A = ah_a = bh_b$ we get $ah_a + a^2 = ah_b + ab$. Thus, $bh_b + a^2 = ah_b + ab$, and so, $(a - h_b)(a - b) = 0$. But $h_b < a$ for an acute-angled triangle, so $a = b$ and cyclic permutation leads to $a = b = c$.

4. In a park there are 10 000 trees, placed in a square lattice of 100 rows and 100 columns. Determine the maximum number of trees that can be cut down satisfying the condition:

sitting on a stump, you cannot see any other stump.

Solution. At most 2500 trees can be cut down and there is a way to chop exactly 2500 trees satisfying the conditions.

Let the trees have integer coordinates from $(0/0)$ to $(99/99)$. We divide the park into 2500 squares containing the trees $(2i/2j)$, $(2i/2j + 1)$, $(2i + 1/2j)$, $(2i + 1/2j + 1)$. Each tree in one of these squares is visible from the other lattice points in the square. Thus only one tree in each square can be cut down.

Let the 2500 trees with coordinates $(2i/2j)$ be cut down $(i, j \in \{0, 1, \dots, 49\})$. For any two lattice points $P(2a/2b)$ and $Q(2c/2d)$ $(a, b, c, d \in \{0, 1, \dots, 49\}, (a/b) \neq (c/d))$, the mid-point R of segment PQ has integer coordinates $(a + c/b + d)$. If one of these coordinates is odd, there is a tree left on R so that P and Q are invisible from each other. If both coordinates are even we can replace Q by R and repeat the same argument. After a finite number of steps we reach a tree. This completes the proof.

That completes the *Skoliad Corner* for this issue. We need good contest materials as well as suggestions for future directions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is

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Shreds and Slices

On Logarithms

It's log! — *The Ren & Stimpy Show*

The logarithm is an important function in mathematics and has some obvious uses, but it appears in some unexpected places as well. We give a diverse list of such applications, but first, we review some definitions.

In the expression $y = b^x$, b is called the *base* and x the *exponent*. With base b fixed, y is a function of x. The inverse function is called the *logarithm*, denoted $\log_b$. Thus, $\log_b y$ is the real number x such that $y = b^x$. For example, $\log_{10} 100 = 2$ and $\log_2 16 = 4$, since $10^2 = 100$ and $2^4 = 16$. The only acceptable bases are $0 < b < 1$ and $b > 1$.

Since the logarithm is based on the exponent function, the main properties of the logarithm are based on the main properties of exponents:

$$
\begin{aligned}
b^0 &= 1\,, && (1)\\
b^1 &= b\,, && (2)\\
b^u b^v &= b^{u+v}\,, && (3)\\
b^u / b^v &= b^{u-v}\,, && (4)\\
(b^u)^c &= b^{uc}\,. && (5)
\end{aligned}
$$

The corresponding properties of logarithms are:

$$
\begin{aligned}
\log_b 1 &= 0\,, && (6)\\
\log_b b &= 1\,, && (7)
\end{aligned}
$$

$$\begin{aligned} \log_b x + \log_b y &= \log_b xy\,, & (8) \\ \log_b x - \log_b y &= \log_b x/y\,, & (9) \\ \log_b x^c &= c \log_b x\,. & (10) \end{aligned}$$

Let us prove these results.

Properties (6) and (7) follow directly from (1) and (2).

Let $u = \log_b x$ and $v = \log_b y$, so $x = b^u$ and $y = b^v$. Then $xy = b^{u+v}$, so $u + v = \log_b xy$. Also, $x/y = b^{u-v}$, so $u - v = \log_b x/y$, proving (8) and (9). Also, $x^c = (b^u)^c = b^{uc}$. Thus $uc = c \log_b x = \log_b x^c$, proving (10).

There is an additional important property, sometimes called the *change of base* formula:

$$\log_b x = \frac{\log_a x}{\log_a b}\,. \tag{11}$$

Since $b^u = x$, $\log_a b^u = \log_a x$. But $\log_a b^u = u \log_a b$, so that $u = \log_b x = \log_a x / \log_a b$. This means that logarithms to different bases are proportional to each other (see Problem 1).

- Base 10 is known as the *common* logarithm, but the most important base is a constant known as $e \approx 2.71828$. Why this constant? Let

  $$f(x) = \int_1^x \frac{1}{t}\,dt\,.$$

  For those not familiar with calculus, you can think of $f(x)$ as being the area under the graph of $y = 1/t$ from $t = 1$ to $t = x$. It turns out that $f(1) = 0$ and $f(x^c) = cf(x)$ for all c and $x > 0$. This implies that f is a logarithm to some base b. We define e to be this base. Hence,

  $$\log_e x = \int_1^x \frac{1}{t}\,dt\,.$$

  Unless otherwise indicated, all following logarithms will now be to base e, called the *natural* logarithm. One formula for e is $e = \lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$.

- $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots\,, \qquad -1 < x \le 1\,.$

  What does the formula become for $x = 1$? What happens as x approaches -1?

- For all $x \ge 0$, $\log(1 + x) \le x$.

- For large n,
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \approx \log n .$$
In fact, the difference approaches a constant:
$$\gamma := \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\log n \approx 0.57722 ,$$
called the *Euler-Mascheroni* constant. It is currently not even known whether γ is irrational or not.

- Stirling's Approximation:
$$\log n! \approx n\log n - n .$$

- For a positive integer N, the number of digits in N expressed in decimal notation is $\lceil \log_{10}(N+1)\rceil$.

- One of the fundamental questions we can ask about prime numbers is how many there are up to a certain number. Let $\pi(n)$ be the number of primes less than or equal to n, and graph $\pi(n)$ in relation to $n/\log n$:

n	$\pi(n)$	$n/\log n$	$\pi(n)/(n/\log n)$
10	4	4.34	0.92
100	25	21.71	1.15
1,000	168	144.76	1.16
10,000	1,229	1,085.74	1.13
100,000	9,592	8,685.89	1.10
1,000,000	78,498	72,382.41	1.08

The agreement, given the irregularity of primes, is good. In fact, the Prime Number Theorem states that the two functions $\pi(n)$ and $n/\log n$ are *asymptotic*; that is, they will continue to approach each other:
$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1 .$$

- We give one final application. In finance circles, there is a rule of thumb called the Rule of 72: If an investment grows at a rate of r% annually, then the number of years it takes for the investment to double is approximately $72/r$.

  For example, suppose that we have a dollar in a bank account, which earns 3% annually, so $r=3$. Assuming that there are no other transactions, at the same time next year, there will be \$1.03 in the account. A year after that there will be $\$1.03 \times 1.03 = \$1.06$ in the account, and in general, after n years, there will be 1.03^n in dollars. The time in years n for the account to double to 2 dollars is given by $1.03^n = 2$.

Taking the logarithm of both sides, $\log 1.03^n = n \log 1.03 = \log 2$, so $n = (\log 2)/(\log 1.03) = 23.44$. The Rule of 72 gives $n \approx 72/3 = 24$, which is fairly close.

In general, we wish to solve for n in the equation

$$\begin{aligned} \left(1+\frac{r}{100}\right)^n &= 2 \\ \Longrightarrow \log\left(1+\frac{r}{100}\right)^n &= n\log\left(1+\frac{r}{100}\right) = \log 2 \\ \Longrightarrow n &= \frac{\log 2}{\log(1+\frac{r}{100})}. \end{aligned}$$

By the power series above,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \approx x$$

for small values of x, so $\log(1 + r/100) \approx r/100$, and

$$n \approx \frac{\log 2}{\frac{r}{100}} = \frac{100\log 2}{r} = \frac{69.17...}{r} \approx \frac{72}{r}.$$

We round $100\log 2$ to 72, because 72 is divisible by many numbers, and so provides for an easier calculation. The approximation is reasonable for many r, as seen in the table below:

r	n	$72/r$
1	69.66	72
2	35.00	36
3	23.45	24
4	17.67	18
6	11.90	12
8	9.01	9
9	8.04	8
12	6.11	6

Problems

1. In a table, list $\log_e n$ and $\log_{10} n$ for various values of n. Show that we can multiply all the values in one list by a constant to obtain all the values in the other list.

2. Before calculators, there was a device called a slide rule which performed numerical calculations. It consists of two, ruled pieces of material which slide against each other. Find out how it works.

Area of a Quadrilateral

The following is a useful, but not too well-known, formula for the area of a quadrilateral:

Let $ABCD$ be a quadrilateral, with sides $a = AB$, $b = BC$, $c = CD$, and $d = DA$. Let K denote the area, and s the semi-perimeter $(a+b+c+d)/2$. Let A also denote the angle at vertex A, etc. Then

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right) .$$

We derive the formula as follows.

By the Cosine Law,

$$BD^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C .$$

Therefore,

$$a^2 - b^2 - c^2 + d^2 = 2ad\cos A - 2bc\cos C ,$$

implying that

$$\begin{aligned}(a^2 - b^2 - c^2 + d^2)^2 &= 4a^2d^2\cos^2 A - 8abcd\cos A\cos C \\ &\qquad + 4b^2c^2\cos^2 C .\end{aligned}$$

Now, the area of $\triangle ABD$ is $\frac{1}{2}ad\sin A$, and the area of $\triangle BCD$ is $\frac{1}{2}bc\sin C$. Thus

$$K = \frac{1}{2}\,ad\sin A + \frac{1}{2}\,bc\sin C .$$

Therefore

$$\begin{aligned}16K^2 &= 4a^2d^2\sin^2 A + 8abcd\sin A\sin C + 4b^2c^2\sin^2 C \\ &= 4a^2d^2\sin^2 A + 8abcd\sin A\sin C + 4b^2c^2\sin^2 C \\ &\quad + 4a^2d^2\cos^2 A - 8abcd\cos A\cos C + 4b^2c^2\cos^2 C \\ &\quad - (a^2 - b^2 - c^2 + d^2)^2 \\ &= 4a^2d^2 + 4b^2c^2 - 8abcd\cos(A+C) - (a^2 - b^2 - c^2 + d^2)^2 \\ &= 4a^2d^2 + 8abcd + 4b^2c^2 - 8abcd\,(\cos(A+C) + 1) \\ &\qquad - (a^2 - b^2 - c^2 + d^2)^2 \\ &= (2ad + 2bc)^2 - (a^2 - b^2 - c^2 + d^2)^2 \\ &\qquad - 16abcd\cos^2\left(\frac{A+C}{2}\right) \\ &= (a^2 + 2ad + d^2 - b^2 + 2bc - c^2) \\ &\qquad \times (b^2 + 2bc + c^2 - a^2 + 2ad - d^2) - 16abcd\cos^2\left(\frac{A+C}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left((a+d)^2-(b-c)^2\right)\left((b+c)^2-(a-d)^2\right) \\
&\qquad -16abcd\cos^2\left(\frac{A+C}{2}\right) \\
&= (-a+b+c+d)(a-b+c+d)(a+b-c+d) \\
&\qquad \times(a+b+c-d) \;-\; 16abcd\cos^2\left(\frac{A+C}{2}\right) \\
&= (2s-2a)(2s-2b)(2s-2c)(2s-2d) \\
&\qquad -16abcd\cos^2\left(\frac{A+C}{2}\right) .
\end{aligned}
$$

Therefore

$$K^2 \;=\; (s-a)(s-b)(s-c)(s-d)-abcd\cos^2\left(\frac{A+C}{2}\right) .$$

An immediate and important corollary of this result is Brahmagupta's Formula: If $ABCD$ is cyclic, then $A+C=180°$, so

$$K^2 \;=\; (s-a)(s-b)(s-c)(s-d)\,.$$

You should now be able to quickly solve Problem 10 on the 1993 Descartes Competition:

Suppose p, q, r, s are fixed real numbers such that a quadrilateral can be formed with sides p, q, r, s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle.

Follow-up to "How to Solve the Cubic"

V.N. Murty writes to recommend the following textbooks as references for the cubic, as well as classic textbooks on algebra:

S. Barnard and J.M. Child, *Algebra*

G. Chrystal, *Algebra*, two volumes

M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 6 of 2001.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <ahchan@fas.harvard.edu>

H273. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, and c be complex numbers such that $a+b+c=0$. Prove that

$$\begin{vmatrix} 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \\ a^2 & c^2 & 2ac-b^2 \end{vmatrix} = 0\,.$$

H274. Find a simplified expression for

$$\sum_{i=1}^{\infty} \frac{i}{k^i}$$

in terms of a real number $k>1$.

H275. How many non-negative integers less than 10^n are there whose digits are in non-increasing order?

H276. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $ABCDE$ be a convex pentagon such that $ACDE$ is a square, and

$$\cot\angle BDE + \cot\angle DEB + \cot\angle EBD = 2\,.$$

Show that $\triangle ABC$ is a right triangle.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A249. *Proposed by Mohammed Aassila, Strasbourg, France.*

A circle is circumscribed around $\triangle ABC$ with sides a, b, c. Let A', B', C' denote the mid-points of the arcs BC, CA, AB, respectively. The straight lines $A'B'$, $B'C'$, $C'A'$ intersect BC and AC, AC and AB, AB and BC, in P, Q, R, S, T, U, respectively. Prove that

$$\frac{[PQRSTU]}{[ABC]} = \frac{(a+b)^2+(b+c)^2+(c+a)^2}{2(a+b+c)^2},$$

where $[X]$ denotes the area of the polygon X.

A250. Suppose polynomial $P(x)$ has integer coefficients such that for any integer m, $P(m)$ is a perfect square. Show that the degree of P is even.

A251. *Proposed by Lee Ho-Joo, undergraduate, Kwangwoon University, Seoul, South Korea.*

In a parallelogram $ABCD$, let P be the intersection of AC with BD. Let M, N be the mid-points of PD, BC, respectively. Prove that the following two statements are equivalent:

(i) $\triangle AMN$ is a (non-degenerate) right-angled triangle such that $AM = MN$.

(ii) Quadrilateral $ABCD$ is a square.

A252. *Proposed by Mohammed Aassila, Strasbourg, France.*

For every positive integer n, prove that there exists a polynomial of degree n with integer coefficients of absolute value at most n, which admits 1 as a root with multiplicity at least $\lfloor \sqrt{n} \rfloor$.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

In Issue 3, **C93** was accidentally printed as **C98**, and there was a typo in the problem. It is corrected here.

C93. Let H be a subset of the positive integers with the property that if x, $y \in H$, then $x + y \in H$. Define the *gap sequence* G_H of H to be the set of positive integers not contained in H.

(a) Prove that if G_H is a finite set, then the arithmetic mean of the integers in G_H is less than or equal to the number of elements in G_H.

(b) Determine all sets H for which equality holds in part (a).

C94. *Proposed by Edward Crane and Russell Mann, graduate students, Harvard University, Cambridge, MA, USA.*

Suppose that V is a k-dimensional vector subspace of the Euclidean space $\mathbb{R}^n$ which is defined by linear equations with coefficients in $\mathbb{Q}$. Let Λ be the lattice in V given by the intersection of V with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and let $\Lambda^\perp$ be the lattice given by the intersection of the perpendicular vector space $V^\perp$ with $\mathbb{Z}^n$. Show that the (k-dimensional) volume of Λ is equal to the (($n-k$)-dimensional) volume of $\Lambda^\perp$. [2000 : 167]

C95. Prove that the curve $x^3 + y^3 = 3xy$ has a horizontal tangent at the origin. (This curve is known as the Folium of Descartes.)

C96. Recall that a *bipartite graph* is a graph whose vertices may be divided into two nonempty disjoint sets (call them L and R, for left and right) so that all of the edges of the graph connect a vertex in L to a vertex in R. In other words, no two vertices in L are joined by an edge, and similarly for R. Let G be a bipartite graph with 27 edges and in which L and R each contain exactly 9 vertices. Show that we can find three vertices l_0, l_1, $l_2 \in L$ and three vertices r_0, r_1, $r_2 \in R$ such that at least six of the nine potential edges l_0r_0, l_0r_1, l_0r_2, l_1r_0, l_1r_1, l_1r_2, l_2r_0, l_2r_1, l_2r_2 are indeed edges of G.

1960 Bulgarian Mathematical Olympiad 1960

3rd Stage

1. Prove that the sum (difference) of two irreducible fractions with different denominators cannot be an integer.

2. Find the maximal and the minimal values of the function

$$y = \frac{x^2 + x + 1}{x^2 + 2x + 2},$$

where x takes all real values.

3. Determine the tangents of x, y, z from the equations $\tan x : \tan y : \tan z = a : b : c$, if $x + y + z = 180^\circ$ and a, b, and c are positive numbers.

4. Two externally tangent circles of radii R and r are given.

(a) Prove that the quadrilateral whose sides are the two common tangents and the chords connecting the points of contact is a trapezoid.

(b) Find the base and the altitude of the trapezoid.

5. The rays a, b, and c have a common origin and do not lie in a plane. The angles $\alpha = \angle(b, c)$, $\beta = \angle(c, a)$, and $\gamma = \angle(a, b)$ are acute and are given in a plane. Construct by ruler and compass the angle between the ray a and the plane which passes through the rays b and c.

6. A sphere is inscribed in a cone. A second sphere tangent to the first is inscribed in the same cone. A third sphere tangent to the second is inscribed in the same cone and so on. Find the sum of the areas of the spheres if the altitude of the cone is equal to h and the angle at the vertex of its plane section through the axes is equal to α.

J.I.R. McKnight Problems Contest 1997

1. Given triangle ABC, with sides a, b, c, where $a + b + c = 60$ and $(b + c)/4 = (a + c)/5 = (a + b)/6$.

   (a) Prove that the sides a, b, c form an arithmetic sequence.

   (b) Find $\sin A : \sin B : \sin C$.

2. (a) Solve:
$$\frac{\log(35 - x^3)}{\log(5 - x)} = 3\,.$$

   (b) Prove:
$$\sum_{r=1}^{n} r \log_{2^r} x = n \log_2 x\,.$$

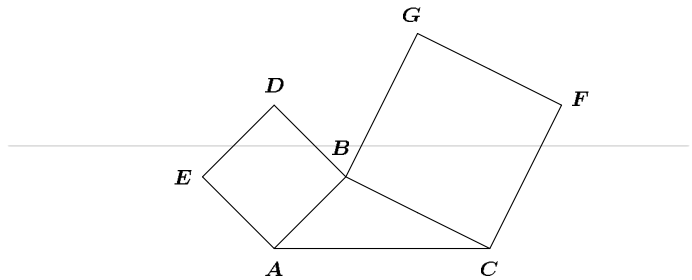
3. Consider three twin brothers: A, A, B, B, C, C. They are to be arranged in a picture in such a way, that no pair of twins will be side by side. Find the number of such arrangements.

4. The natural numbers are arranged in diamonds as shown below. Conjecture and prove a formula for the sum of the numbers in the n^{th} diamond.

$$\begin{array}{c} 1 \end{array} \qquad \begin{array}{c} 1 \\ 2\quad 2 \\ 3 \end{array} \qquad \begin{array}{c} 1 \\ 2\quad 2 \\ 3\quad 3\quad 3 \\ 4\quad 4 \\ 5 \end{array} \qquad \begin{array}{c} 1 \\ 2\quad 2 \\ 3\quad 3\quad 3 \\ 4\quad 4\quad 4\quad 4 \\ 5\quad 5\quad 5 \\ 6\quad 6 \\ 7 \end{array}$$

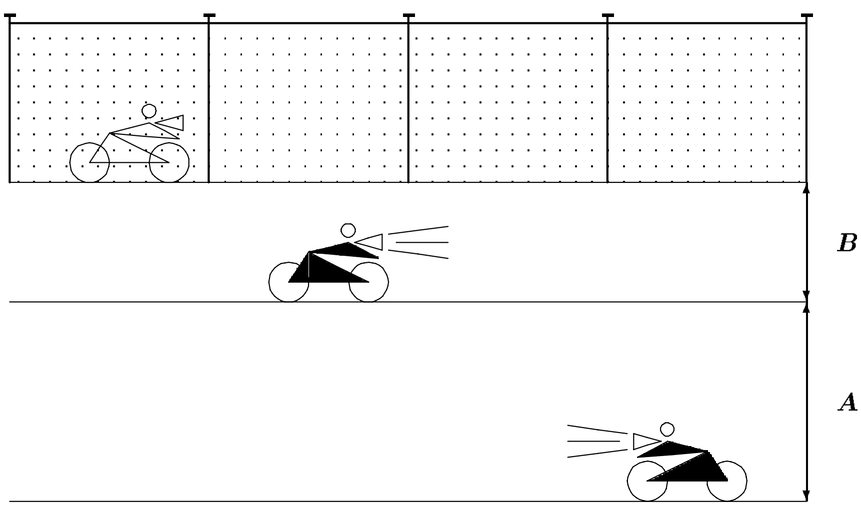
5. Find the limits N, M for which the inequality
$$N < \frac{(x + 1)(x + 4)}{x} < M$$
is satisfied by no real values of x.

6. Squares $ABDE$ and $BCFG$ are drawn outwards on the sides of $\triangle ABC$. Prove that AC is parallel to DG if and only if $\triangle ABC$ is isosceles.



7. Given $f(x+y) + f(x-y) = 2f(x)\cos y$, $f(0) = a$, and $f(\pi/2) = b$. Find $f(t)$.

8. Two motorcycles are approaching each other at night on a straight, two-lane highway. Each vehicle is travelling in the centre of its lane and the centres of the 2 lanes are A metres apart. The eastbound cycle is travelling at M metres per second. The westbound cycle is travelling at a rate of N metres per second, and its headlight casts a shadow of the eastbound cycle onto a fence, B metres from the centre of the eastbound lane. How fast is the shadow of the eastbound cycle moving on the fence? (Express your answer in terms of A, B, M, and N).



Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Prove that

$$\frac{1}{1999} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998} < \frac{1}{44}\,.$$

(1997 CMO, Problem 3)

Solution. For the left inequality,

$$\begin{aligned}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998} &= \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{1997}{1996}\cdot\frac{1}{1998}\\ &> 1\cdot 1\cdot 1\cdots 1\cdot\frac{1}{1998}\\ &= \frac{1}{1998} > \frac{1}{1999}\,.\end{aligned}$$

For the right inequality, let

$$P = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{1997}{1998}\,.$$

Note that P is positive. Then,

$$\begin{aligned}P &< \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{1998}{1999}\\ &= \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{1998}{1997}\cdot\frac{1}{1999}\\ &= \frac{1}{1999P}\,,\end{aligned}$$

so that

$$P^2 < \frac{1}{1999} < \frac{1}{1936} = \frac{1}{44^2}\,,$$

which implies that $P < 1/44$. Thus,

$$\frac{1}{1999} < P < \frac{1}{44}\,.$$

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"$\times$11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 March 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2551. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Suppose that a_k $(1 \leq k \leq n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

Prove that

$$\prod_{j=1}^{n} \sum_{k=1}^{n} e_{j,k} a_k^2 \;\geq\; \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2 .$$

2552. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Suppose that a, b, $c > 0$. If $x \geq \frac{a+b+c}{3\sqrt{3}} - 1$, prove that

$$\frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} \;\geq\; abc\,.$$

2553. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find all real roots of the equation

$$\frac{\left(\sqrt{2x^2-2x+12} - \sqrt{x^2-5}\right)^3}{(5x^2-2x-3)\sqrt{2x^2-2x+12}} \;=\; \frac{2}{9}\,.$$

2554. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In triangle ABC, prove that at least one of the quantities

$$(a+b-c)\tan^2\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right),$$
$$(-a+b+c)\tan^2\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right),$$
$$(a-b+c)\tan^2\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right),$$

is greater than or equal to $\frac{2r}{3}$, where r is the radius of the incircle of $\triangle ABC$.

2555. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In any triangle ABC, show that

$$\sum_{\text{cyclic}} \frac{1}{\tan^3\frac{A}{2}+\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^3} < \frac{4\sqrt{3}}{3}.$$

2556*. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

A lattice point is called *visible* (from the origin) if its coordinates are coprime numbers. Is there any lattice point whose distance from each visible lattice point is at least 2000?

2557. *Proposed by Gord Sinnamon, University of Western Ontario, London, Ontario, and Hans Heinig, McMaster University, Hamilton, Ontario.*

(a) Show that for all positive sequences $\{x_i\}$ and all integers $n>0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \le 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}.$$

(b)* Does the above inequality remain true without the factor 2?
(c)* [Proposed by the editors] What is the minimum constant c that can replace the factor 2 in the above inequality?

2558. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let Z be a half-plane bounded by a line L. Let A, B and C be any three points on L such that C lies between A and B. Denote the three semicircles in Z on AB, AC and CB as diameters by K_0, K_1 and K_2 respectively. Let F be the family of semicircles in Z with diameters on L (including all half-lines in Z perpendicular to L). Denote by f_{XY} the unique semicircle passing through the pair of distinct points X, Y in $Z\cup L$. Let P, Q, R, be three points on K_2, K_1, K_0, respectively.

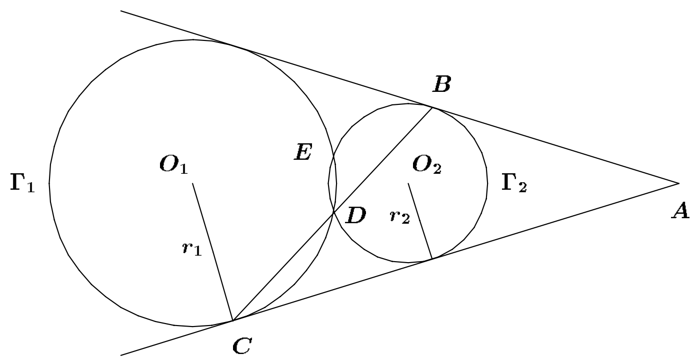
If f_{AP}, f_{BQ} and f_{CR} concur at T, and the lines AP, BQ, CR concur at S, prove that f_{AP}, f_{BQ} and f_{CR} are orthogonal to K_2, K_1, K_0, respectively, and that the circle PQR is tangent to each semicircle K_j, $(j = 0, 1, 2)$.

2559 *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Triangle ABC has incentre I. Show that $CA + AI = CB$ if and only if $\angle CAB = 2\angle ABC$.

2560★ *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Lines AB and AC are common tangents to the circles Γ_1 and Γ_2 with distinct radii r_1 and r_2 respectively, as shown.



B is a point of tangency on Γ_2 and C is a point of tangency on Γ_1. The intersection points of the circles, D and E, exist, CDB is a straight line, and $CD = DB$.

Construct such a figure using straightedge and compass.

2561. *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let M disks from N different colours be placed in a row such that k_i disks are from the i^{th} colour $(i = 1, 2, \ldots, N)$ and $k_1 + k_2 + \cdots + k_N = M$.

A move is an exchange of two adjacent disks.

Determine the smallest number of moves needed to rearrange the row such that all disks of the same colour are adjacent to one another.

2562. *Proposed by Bernardo Recamán Santos, Colegio Hacienda Los Alcaparros, Bogotá, Colombia.*

(a) Show that for all sufficiently large n, it is possible to find a set of n (not necessarily distinct) positive integers whose sum is the square root of their product.

(b)★ Are there infinitely many n for which there is a unique set of n numbers with property (a)?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The name of O. Ciaurri, Universidad de La Roija, Logroño, Spain, a co-solver with Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain, was unfortunately omitted in the solution to problem 2404. Our apologies.

2436. [1999: 173, 191] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] \;=\; 0\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and by Heinz-Jürgen Seiffert, Berlin, Germany (combined by the editor).

Correction: instead of

$$t^p + t^{-p} + 2^p \;=\; t + t^{-1} + 2\,.$$

read

$$t^p + t^{-p} + 2^p \;=\; \left(t + t^{-1}\right)^p + 2\,.$$

2451. [1999: 306] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Construct an infinite sequence, $\{A_n\}$, of infinite subsets of $\mathbb{N}$ with the following properties:

(a) the intersection of any two distinct sets A_n and A_m is a singleton;

(b) the singleton in (a) is a different one if at least one of the distinct sets A_n, A_m, is changed (so the new pair is again distinct);

(c) every natural number is the intersection of (exactly) one pair of distinct sets as in (a).

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

Arrange all the natural numbers in an infinite triangular array $\{a_{ij}\}$, $1 \le i \le j$, filling column n with n consecutive integers, as follows:

$$\begin{array}{cccccccc}
1 & 2 & 4 & 7 & 11 & 16 & \dots \\
 & 3 & 5 & 8 & 12 & 17 & \dots \\
 & & 6 & 9 & 13 & 18 & \dots \\
 & & & 10 & 14 & 19 & \dots \\
 & & & & 15 & 20 & \dots \\
 & & & & & 21 & \dots \\
 & & & & & & \ddots
\end{array}$$

Then define:

$$\begin{array}{rcll}
A_0 & = & \{1, 3, 6, 10, 15, \dots\} & \text{(main diagonal)} \\
A_1 & = & \{1, 2, 4, 7, 11, 16, \dots\} & \text{(first row)} \\
A_2 & = & \{2, 3, 5, 8, 12, 17, \dots\} & \text{(union of column 2 and row 2)} \\
 & & \dots & \\
A_n & = & \text{union of column } n \text{ and row } n & \\
 & & \dots &
\end{array}$$

Since $A_0 \cap A_i = \{a_{ii}\}$ for all $1 \leq i$ and $A_i \cap A_j = \{a_{ij}\}$ for all $1 \leq i < j$, it is clear that (a), (b) and (c) hold.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER and DOUG CASHING, St. Bonaventure University, St. Bonaventure, NY, USA; DEE SNELL, University of Wisconsin, La Crosse, WI, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Dergiades, Sealy, Sedinger and Cashing, Snell, and Young all gave solutions similar to the above.

2452. [1999: 307, 428] *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!}.$$

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^4}{(n+1)!}.$$

(c) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^6}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^7}{(n+1)!}.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA. (modified slightly by the editor).

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2n+1}{(2n)!} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n)!}$

$= \sinh 1 + \cosh 1 = e$, while

$$\sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!} = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

$$= \cosh 1 + \sinh 1 = e.$$

(b) and (c). For $k \geq 1$, let

$$I_k = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^k}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^{k-1}}{(n)!}.$$

It can be shown easily by the Ratio Test that I_k converges absolutely for all $k \geq 1$.

Note that $I_1 = \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} = \frac{1}{e} - 1$, while for $k \geq 2$,

$$I_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{i=0}^{k-1} \binom{k-1}{i} n^i = \sum_{i=0}^{k-1} \binom{k-1}{i} \sum_{n=1}^{\infty} \frac{(-1)^n n^i}{n!}$$

$$= \sum_{i=0}^{k-1} \binom{k-1}{i} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)^i}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} - \sum_{i=1}^{k-1} \binom{k-1}{i} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^i}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} - \sum_{i=1}^{k-1} \binom{k-1}{i} (1 + I_i)$$

$$= \left(\frac{1}{e} - 1\right) - \sum_{i=1}^{k-1} \binom{k-1}{i} - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i$$

$$= \left(\frac{1}{e} - 1\right) - (2^{k-1} - 1) - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i$$

$$= \frac{1}{e} - 2^{k-1} - \sum_{i=1}^{k-1} \binom{k-1}{i} I_i .$$

Define $\{a_k\}$ by $a_1 = 1$ and $a_k = 1 - \sum_{i=1}^{k-1} \binom{k-1}{i} a_i$ for $k \geq 2$. We show by induction that $I_k = \frac{a_k}{e} - 1$ for all $k \geq 1$.

Since $I_1 = \frac{1}{e} - 1$, the claim is true for $k = 1$.

Suppose $I_k = \frac{a_k}{e} - 1$ for some $k \geq 1$. Then using the recurrence relation obtained above, we have

$$\begin{aligned} I_{k+1} &= \frac{1}{e} - 2^k - \sum_{i=1}^{k} \binom{k}{i} I_i = \frac{1}{e} - 2^k - \sum_{i=1}^{k} \binom{k}{i} \left(\frac{a_i}{e} - 1\right) \\ &= \frac{1}{e} - 2^k - \frac{1}{e} \sum_{i=1}^{k} \binom{k}{i} a_i + \sum_{i=1}^{k} \binom{k}{i} \\ &= \frac{1}{e} - 2^k - \frac{1}{e}\left(1 - a_{k+1}\right) + \left(2^k - 1\right) = \frac{a_{k+1}}{e} - 1, \end{aligned}$$

completing the induction.

The first ten terms of the sequence $\{a_k\}$ are:

$$1,\ 0,\ -1,\ -1,\ 2,\ 9,\ 9,\ -50,\ -167,\ -513,\ \cdots .$$

Hence it follows immediately that $I_3 = \frac{-1}{e} - 1 = I_4$ and $I_6 = \frac{9}{e} - 1 = I_7$.

Also solved by MICHEL BATAILLE, Rouen, France; THE BOOKERY PROBLEM GROUP, Walla Walla, WA, USA ; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); and the proposer.

Actually, as pointed out by Janous, all these results were contained in the proposer's paper "Apropos Bell and Stirling Numbers" *which appeared in* ***CRUX with MAYHEM*** [1999 : 274-281].

Diminnie commented that it would be interesting to know whether there are any values $k \geq 7$, such that $I_k = I_{k+1}$ and in general, whether there are distinct values m, n, $m > n \geq 7$ such that $I_m = I_n$. He conjectured that there are none, but was not able to come up with a proof.

2453. [1999: 307, 428] *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^3}{(2n+1)!} = -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\,.$$

(b) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^3}{(2n)!} = -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\,.$$

(c) $$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!}\right)^2+\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!}\right)^2 = 2\,.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

(a) $$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^3}{(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n)!}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{2n(2n-1)}{(2n)!}+3\sum_{n=0}^{\infty}(-1)^n\frac{2n}{(2n)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\\ &= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!}+3\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}+3\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\\ &= -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!} = -3\sin 1\,.\end{aligned}$$

(b) $$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^3}{(2n)!} &= \sum_{n=1}^{\infty}(-1)^n\frac{(2n)^2}{(2n-1)!}\\ &= \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)(2n-2)}{(2n-1)!}+3\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)}{(2n-1)!}\\ &\quad+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}\\ &= \sum_{n=2}^{\infty}(-1)^n\frac{1}{(2n-3)!}+3\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!}+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}\\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)!}+3\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}\end{aligned}$$

$$= -3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!} = -3\cos 1\,.$$

(c) $$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{(2n)!}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2n}{(2n)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$$
$$= -\sin 1+\cos 1\,,$$

and

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{2n}{(2n-1)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{(2n-1)!}+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)!}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-2)!}+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}$$
$$= -\cos 1-\sin 1\,.$$

Therefore,

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^2}{(2n)!}\right)^2$$
$$= (-\sin 1+\cos 1)^2+(-\cos 1-\sin 1)^2$$
$$= 2\,.$$ ■

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NIKOLAOS DERGIADES, Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, WA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); and the proposer.

Janous pointed out that, as in Problem 2452, the identities in this problem all follow readily from results established in the proposer's paper Apropos Bell and Stirling Numbers. [*Ed: See comments at the end of the solution to Problem 2452.*] *More specifically,* (a) *holds if and only if* $c_3 \sin 1 + d_3 \cos 1 = -3(c_0 \sin 1 + d_0 \cos 1)$ *and* (b) *holds if and only if* $c_3 \cos 1 - d_3 \sin 1 = -3(c_0 \cos 1 - d_0 \sin 1)$. *These are true since* $c_0 = 1$, $c_3 = -3$ *and* $d_0 = d_3 = 0$. [*Ed: See Table 1 on p. 276 of the proposer's paper.*]

As for (c), *we have, more generally, that*

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^k}{(2n+1)!}\right)^2+\left(\sum_{n=0}^{\infty}(-1)^n\frac{(2n)^k}{(2n)!}\right)^2 = (c_k\sin 1+d_k\cos 1)^2+(c_k\cos 1-d_k\sin 1)^2 = c_k^2+d_k^2\,.$$

[*Ed: In particular, for* $k=2$ *we get the answer* 2 *since* $c_2=-1$ *and* $d_2=1$. *For* $k=3$ *and* 4 *we would get the answers* 9 *and* 61 *respectively. Hess commented that he determined these two values "from the misprinted problem"*].

2454. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three circles intersect each other orthogonally at pairs of points A and A', B and B', and C and C'. Prove that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

I. *Solution by Michel Bataille, Rouen, France.*

Among the three given circles, let Γ be the one that does not pass through A, and let I denote the inversion with centre A such that $I(\Gamma)=\Gamma$. Because of the mutual orthogonality of the three circles, I transforms the two other circles into two perpendicular lines that are orthogonal to $I(\Gamma)=\Gamma$.

Hence these two lines are diameters of Γ, meeting Γ at $I(B)$, $I(B')$, and $I(C)$, $I(C')$, respectively.

Clearly, the quadrilateral with vertices $I(B)$, $I(C)$, $I(B')$ and $I(C')$ is a rectangle, so that the lines through $I(B)$, $I(C)$, and through $I(B')$, $I(C')$, are parallel. Hence the images of these lines under I are circles, tangent at A. But these circles are precisely the circumcircles of $\triangle ABC$ and $\triangle AB'C'$, and so we are done.

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let the three circles be Γ_1, Γ_2 and Γ_3, and let A and A' be the intersections of Γ_1 and Γ_2, let B and B' be the intersections of Γ_1 and Γ_3, and let C and C' be the intersections of Γ_2 and Γ_3. See Figure 1.

We assume that Γ_3 intersects Γ_1 and Γ_2 orthogonally.

Let O_3 be the centre of Γ_3. Then O_3B and O_3B' are tangent to Γ_1 at B and B' respectively, and O_3C and O_3C' are tangent to Γ_2 at C and C' respectively. Note that O_3 lies on the line AA'.

Hence we have

$$\angle CAA' = \angle O_3CA' \qquad \text{and} \qquad \angle B'AA' = \angle O_3B'A'\,.$$

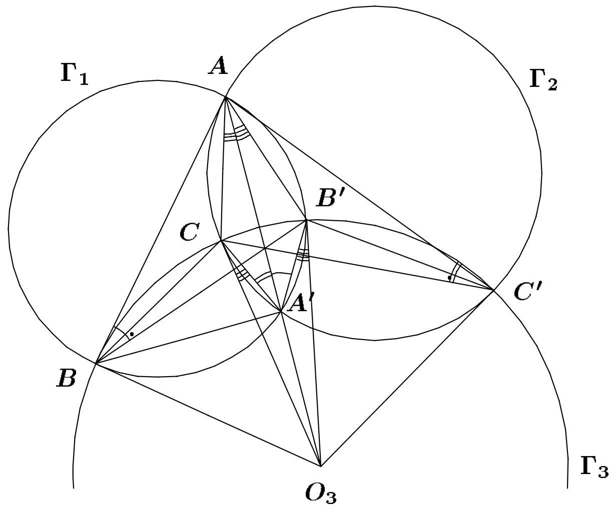


Figure 1.

It follows that

$$\begin{aligned} \angle CAB' &= \angle CAA' + \angle B'AA' = \angle O_3CA' + \angle O_3B'A' \\ &= \angle CA'B' - \angle CO_3B' . \end{aligned} \qquad (1)$$

Since O_3 is the centre of Γ_3, we have

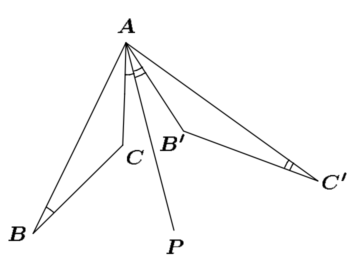
$$\angle CBB' = \angle CC'B = \tfrac{1}{2}\angle CO_3B' .$$

Since $\angle ABB' = \angle AA'B$ and $\angle AC'C = \angle AA'C$, we get

$$\begin{aligned} \angle ABC + \angle AC'B' &= (\angle ABB' - \angle CBB') + (\angle AC'C - \angle CC'B') \\ &= (\angle ABB' + \angle AC'C) - (\angle CBB' + \angle CC'B') \\ &= (\angle AA'B' + \angle AA'C) - \angle CO_3B' \\ &= \angle CA'B' - \angle CO_3B' . \end{aligned}$$

Thus we have, from (1),

$$\angle CAB' = \angle ABC + \angle AC'B' . \qquad (2)$$



Let P be an interior point of $\angle CAB'$ such that

$$\angle CAP = \angle ABC . \qquad (3)$$

Then we obtain from (2)

$$\angle B'AP = \angle AC'B' . \qquad (4)$$

From (3) and (4), we have that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ are tangent to AP at A. Thus, the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

Comment. As shown in the proof, the condition of the orthogonality of Γ_1 and Γ_2 is not necessary.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

All solvers other than Seimiya made use of inversion.

2455. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three equal circles, centred at A, B and C intersect at a common point P. The other intersection points are L (not on circle centre A), M (not on circle centre B), and N (not on circle centre C). Suppose that Q is the centroid of $\triangle LMN$, that R is the centroid of $\triangle ABC$, and that S is the circumcentre of $\triangle LMN$.

(a) Show that P, Q, R and S are collinear.

(b) Establish how they are distributed on the line.

Solution by Ho-joo Lee, Seoul, South Korea (modified by the editor).

Let $AB \bigcap PN = Z$, $BC \bigcap PL = X$, and $CA \bigcap PM = Y$. Since $\overline{AP} = \overline{AN} = \overline{BP} = \overline{BN}$, $PANB$ is a rhombus whose diagonals AB and PN are perpendicular and bisect one another (so that Z is the mid-point of both $\overline{AB}$ and $\overline{PN}$). Similarly for $BC \perp PL$ and $CA \perp PM$. Since $\triangle XYZ$ is the mid-point triangle of $\triangle ABC$ (with sides $XY || AB$, etc.), we have $ZP \perp XY$, $XP \perp YZ$, and $YP \perp ZX$, so that P is the orthocentre of $\triangle XYZ$. Moreover, R (the centroid of $\triangle ABC$) is the centroid of $\triangle XYZ$. Let T denote the circumcentre of $\triangle XYZ$. Thus P, R, and T lie on the Euler line of $\triangle XYZ$ and satisfy

$$\overline{PR} = 2\overline{RT}. \tag{1}$$

The dilatation $P\left(\frac{1}{2}\right)$ not only carries $\triangle LMN$ to $\triangle XYZ$, but its centroid Q to R — so that

$$\overline{PQ} = 2\overline{PR}, \tag{2}$$

and its circumcentre S to T — so that

$$\overline{QS} = 2\overline{RT}. \tag{3}$$

Since P, R, and T are collinear, so are P, Q, R, and S. Finally, we have $\overline{PR} = \overline{RQ}$ (from (2)) and $\overline{PR} = \overline{QS}$ (from (1) and (3)), so that the points P, R, Q, and S are equally spaced in this order along the line.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; CHOONGYUP SUNG, Pusan, Korea; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

An immediate consequence of the featured proof is that P (the orthocentre of $\triangle XYZ$) is also the orthocentre of $\triangle LMN$, and that the circle LMN is congruent to the three given circles. Since $PLMN$ forms an orthocentric quadrangle (meaning each point is the orthocentre of the triangle formed by the other three), the configuration of the congruent circumcircles of these four triangles has a long history. The converse — that the circle LMN is congruent to the original 3 — can be traced back to R.A. Johnson [A circle theorem, American Math. Monthly **23** *(1916), 161-162;* see also *Arnold Emch,* Remarks on the foregoing circle theorem, **23** *(1916), 162-164], and to G. Ţiţeica (for whom the editor has never seen a specific reference, so does not know what he (or she) did and when it was done). This configuration was Dana McKenzie's starting point; his work* [Triquetras and porisms, *College Math. J.* **23** *2 (March, 1992), 118-131] would certainly be of interest to* ***CRUX with MAYHEM*** *readers.*

2456. [1999: 307] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect orthogonally at P. A third circle touches them at Q and R. Let X be any point on this third circle. Prove that the circumcircles of $\triangle XPQ$ and $\triangle XPR$ intersect at $45°$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let P_1 be the second point of intersection of the circles which cut orthogonally.

Orthogonality invites inversion, so invert with respect to the circle with centre P and radius PP_1.

The orthogonal circles invert into a pair of mutually perpendicular lines through P_1, and the third circle becomes the circle tangent to these lines at the maps, Q' and R', of Q and P. That is to say, the inverse of the third circle is a circle on the chord $Q'R'$ which contains a right angle. Clearly, this circle passes through X', the inverse of the point X.

The circumcircles of $\triangle XOQ$ and $\triangle XPR$ become the straight lines through $X'Q'$ and $X'R'$ respectively.

Since an inscribed angle equals, in degrees, half of its intercepted arc, we clearly have

$$\angle Q'X'R' \;=\; \tfrac{90°}{2} \;=\; 45° \qquad \text{(Figure 1)}$$

or $$\angle Q'X'R' \;=\; \tfrac{270°}{2} \;=\; 135° \;=\; 180° - 45° \qquad \text{(Figure 2)}$$

and the conclusion follows.

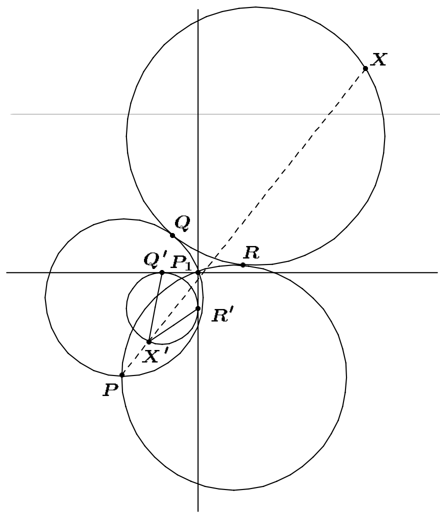


Figure 1

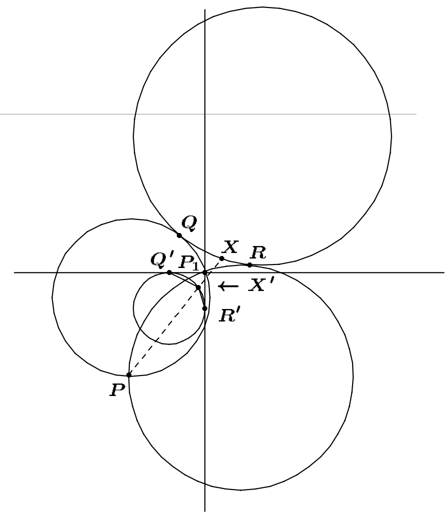


Figure 2

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

All solvers used inversion, except for Arslanagić, Seimiya and Smeenk, who used direct Euclidean methods.

2457. [1999: 308] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In quadrilateral $ABCD$, we have $\angle A + \angle B = 2\alpha < 180°$, and $BC = AD$. Construct isosceles triangles DCI, ACJ and DBK, where I, J and K are on the other side of CD from A, such that $\angle ICD = \angle IDC = \angle JAC = \angle JCA = \angle KDB = \angle KBD = \alpha$.

(a) Show that I, J and K are collinear.

(b) Establish how they are distributed on the line.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Since $\triangle ICD \sim \triangle JCA$, we have $\triangle JIC \sim \triangle ADC$ [by side-angle-side], so that $\angle JIC = \angle ADC$. Similarly, from $\triangle ICD \sim \triangle KBD$, we have $\triangle DIK \sim \triangle DCB$ and $\angle DIK = \angle DCB$. Therefore, $\angle JIK = \angle JIC + \angle DIK - \angle CID = (\angle ADC + \angle DCB) - \angle CID = (2\pi - 2\alpha) - (\pi - 2\alpha) = \pi$ [since in $ABCD$, $\angle A + \angle B = 2\alpha$, while $\triangle CID$ has base angles equal to α]. Consequently, JIK is a straight line.

Moreover, $\dfrac{IJ}{IC} = \dfrac{AD}{CD} = \dfrac{BC}{CD} = \dfrac{IK}{ID} = \dfrac{IK}{IC}$. Therefore, $IJ = IK$, or I is the mid-point of JK. *QED*.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let M be the matrix that represents the rotation about the origin through angle $180^\circ - 2\alpha \neq 0$. So

$$M\cdot \overrightarrow{JA} = \overrightarrow{JC}\ , \quad M\cdot \overrightarrow{ID} = \overrightarrow{IC} \quad M\cdot \overrightarrow{KD} = \overrightarrow{KB}\ , \quad M\cdot \overrightarrow{DA} = \overrightarrow{CB}\ ,$$

and hence

$$\begin{aligned} M\cdot \overrightarrow{IJ} &= M\cdot\left(\overrightarrow{ID} + \overrightarrow{DA} + \overrightarrow{AJ}\right) = M\cdot \overrightarrow{ID} + M\cdot \overrightarrow{DA} - M\cdot \overrightarrow{JA} \\ &= \overrightarrow{IC} + \overrightarrow{CB} - \overrightarrow{JC} = \overrightarrow{IB} - \left(\overrightarrow{IC} - \overrightarrow{IJ}\right) \\ &= \overrightarrow{IB} - \overrightarrow{IC} + \overrightarrow{IJ}\ . \\ M\cdot \overrightarrow{IK} &= M\cdot\left(\overrightarrow{ID} + \overrightarrow{DK}\right) = M\cdot \overrightarrow{ID} - M\cdot \overrightarrow{KD} = \overrightarrow{IC} - \overrightarrow{KB} \\ &= \overrightarrow{IC} - \overrightarrow{IB} + \overrightarrow{IK}\ . \end{aligned}$$

By addition, $M\cdot\left(\overrightarrow{IJ} + \overrightarrow{IK}\right) = \overrightarrow{IJ} + \overrightarrow{IK}$, which implies that $\overrightarrow{IJ} + \overrightarrow{IK} = \overrightarrow{0}$, which means that the points I, J, K are collinear and that I is the mid-point of the segment JK.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece (a second solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2458. [1999: 308] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $ABCD$ be a quadrilateral inscribed in the circle centre O, radius R, and let E be the point of intersection of the diagonals AC and BD. Let P be any point on the line segment OE and let K, L, M, N be the projections of P on AB, BC, CD, DA respectively.

Prove that the lines KL, MN, AC are either parallel or concurrent.

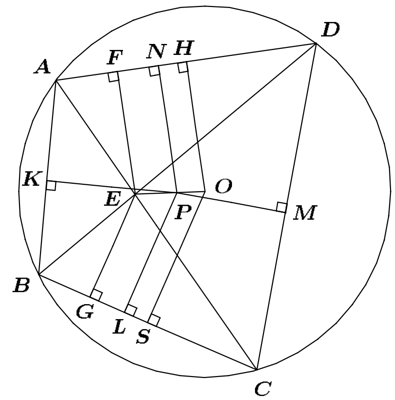
Solution by Toshio Seimiya, Kawasaki, Japan.

Let F and G be the feet of the perpendiculars from E to AD and BC, respectively. The triangles EAD and EBC are similar, because

$$\angle EAD = \angle CAD = \angle DBC = \angle EBC$$

and

$$\angle AED = \angle BEC\ .$$



Since $EF \perp AD$ and $EG \perp BC$, then $\triangle EAF$ is similar to $\triangle EBG$ and $\triangle EDF$ is similar to $\triangle ECG$. Hence $AF : BG = EF : EG = FD : GC$ and therefore,

$$AF : FD \;=\; BG : GC\,. \tag{1}$$

Let H and S be the feet of the perpendiculars from O to AD and BC, respectively. Since O is the centre of the circle, $AH = HD$ and $BS = SC$, giving

$$AH : HD \;=\; BS : SC\,. \tag{2}$$

Since EF, PN and OH are perpendicular to AD, then $EF \parallel PN \parallel OH$, and therefore, $FN : NH = EP : PO$. Similarly, $GL : LS = EP : PO$, so that

$$FN : NH \;=\; GL : LS\,. \tag{3}$$

From (1), (2) and (3) we obtain

$$AN : ND \;=\; BL : LC\,; \tag{4}$$

denote these two ratios by p. Similarly,

$$AK : KB \;=\; DM : MC\,; \tag{5}$$

denote these two ratios by q.

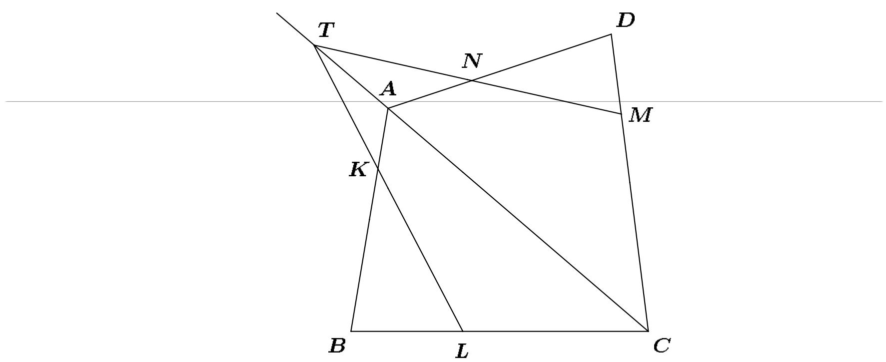
If $p = q$, then, by Thales' Theorem, $KL \parallel AC$ and $MN \parallel AC$.

If $p \neq q$, let T be the intersection of KL and AC. By Menelaus' Theorem,

$$\frac{AT}{TC}\cdot\frac{CL}{LB}\cdot\frac{BK}{KA} \;=\; 1\,.$$

Using (4) and (5), we obtain

$$\frac{AT}{TC}\cdot\frac{DN}{NA}\cdot\frac{CM}{MD} \;=\; 1\,.$$



It follows (by the converse of Menelaus' Theorem) that T, M and N are collinear. Therefore, KL, MN and AC are concurrent at T.

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2459. [1999: 308] *Proposed by Vedula N. Murty, Visakhapatnam, India, modified by the editors.*

Let P be a point on the curve whose equation is $y = x^2$. Suppose that the normal to the curve at P meets the curve again at Q. Determine the minimal length of the line segment PQ.

Solution by Michel Bataille, Rouen, France.

Let $P = (t, t^2)$ be a point on the parabola with $t \neq 0$. Then $Q = (u, u^2)$, where $u \neq 0$ and $u \neq t$. The slope of the tangent line at P is $2t$, so the slope of the normal at P is $-\frac{1}{2t}$; that is, $\frac{u^2-t^2}{u-t} = -\frac{1}{2t}$. Hence, $u = -t - \frac{1}{2t}$, and

$$\begin{aligned} PQ^2 &= (t-u)^2 + (t^2-u^2)^2 = (t-u)^2[1+(t+u)^2] \\ &= \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right) = \left(4t^2 + \frac{1}{4t^2} + 2\right)\left(1 + \frac{1}{4t^2}\right) . \end{aligned}$$

Substituting $z = \frac{1}{4t^2} > 0$, we obtain

$$\begin{aligned} PQ^2 &= \left(z^2 + \frac{1}{z^2} + 2\right)\left(1 + \frac{1}{z^2}\right) \\ &= z^2 + 3z + 3 + \frac{1}{z} = \left(z - \frac{1}{2}\right)^2 \left(1 + \frac{4}{z}\right) + \frac{27}{4} . \end{aligned}$$

Hence $PQ^2 \geq \frac{27}{4}$ with equality if and only if $z = \frac{1}{2}$. Therefore, the minimal length of the segment PQ is $\frac{3\sqrt{3}}{2}$ obtained when P is either $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ or $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ELSIE CAMPBELL, Angelo State University, San Angelo, TX, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark (two solutions); NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; KARTHIK GOPALRATNAM, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; DAVID VELLA, Skidmore College, Saratoga Springs, NY, USA; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer. There was also one incorrect solution submitted.

Klamkin noted that this problem has appeared as # 11 in the first William Lowell Putnam Mathematical Competition [1]. *Most of the solvers used Calculus to find the minimum. Dergiades, Leversha and Sung gave "No Calculus" solutions.*

[1] *A.M. Gleason, R.E. Greenwood and L.M. Kelly, The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1938-1964, M.A.A., Washington, D.C., 1980, pp. 5, 91.*

Congratulations, Paco Bellot!

The International Federation of Mathematics Competitions has awarded the Paul Erdős Prize to Francisco Bellot Rosado in recognition of his work, since 1988, with Olympiad students. The prize will be presented in August 2000 during the International Conference of Mathematics Education in Japan. Francisco Bellot Rosado is the first Spaniard to receive this prize, and is the first High School teacher in the world to have been awarded it.

Congratulations, Paco Bellot (as he is known to his students and colleagues.)