SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1637. [1991: 114; 1992: 125; 1994: 165] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

Comment by Waldemar Pompe, student, University of Warsaw, Poland.

We can use ***Crux* 2015** [1998: 305] to derive the strengthening of ***Crux* 1637** given on [1994: 165], namely that

$$\sum \frac{\sin B + \sin C}{A} \geq \frac{9\sqrt{3}}{\pi}$$

for *all* triangles ABC.

Without loss of generality, we can assume that $A \geq B \geq C$. Then

$$\frac{1}{A} \leq \frac{1}{B} \leq \frac{1}{C}$$

and

$$\sin B + \sin C \leq \sin A + \sin C \leq \sin A + \sin B\,.$$

By Chebyshev's Inequality and ***Crux* 2015**, we have

$$3\sum \frac{\sin B + \sin C}{A} \geq \left(2\sum \sin A\right) \cdot \left(\sum \frac{1}{A}\right) \geq \frac{27\sqrt{3}}{\pi}\,,$$

and the result follows.

2257. [1997: 300] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at the point O. Let OK, OL, OM, ON, be the altitudes of triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle DAO$, respectively.

Prove that if $OK = OM$ and $OL = ON$, then $ABCD$ is a parallelogram.

I. *Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

Let $OK = OM = h$, $OA = a$, $OB = b$, $OC = c$, $OD = d$, and $\angle AOB = v$. Expressing in two ways the area of $\triangle AOB$, we get

$$\tfrac{1}{2}AB \cdot h = \tfrac{1}{2}ab\sin v\,,$$

and so

$$\frac{1}{h} \;=\; \frac{\sqrt{a^2+b^2-2ab\cos v}}{ab\sin v} \;=\; \frac{1}{\sin v}\sqrt{\frac{1}{b^2}+\frac{1}{a^2}-2\cdot\frac{1}{b}\cdot\frac{1}{a}\cdot\cos v}\;.$$

Similarly,

$$\frac{1}{h} \;=\; \frac{1}{\sin v}\sqrt{\frac{1}{d^2}+\frac{1}{c^2}-2\cdot\frac{1}{d}\cdot\frac{1}{c}\cdot\cos v}\;,$$

so that

$$\sqrt{\frac{1}{b^2}+\frac{1}{a^2}-2\cdot\frac{1}{b}\cdot\frac{1}{a}\cdot\cos v} \;=\; \sqrt{\frac{1}{d^2}+\frac{1}{c^2}-2\cdot\frac{1}{d}\cdot\frac{1}{c}\cdot\cos v}\;, \qquad (1)$$

and similarly,

$$\sqrt{\frac{1}{a^2}+\frac{1}{d^2}+2\cdot\frac{1}{a}\cdot\frac{1}{d}\cdot\cos v} \;=\; \sqrt{\frac{1}{b^2}+\frac{1}{c^2}+2\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\cos v}\;. \qquad (2)$$

Now, consider another convex quadrilateral $A_1B_1C_1D_1$, with diagonals intersecting in O_1, and such that $\angle A_1O_1B_1 = v$, $O_1A_1 = \frac{1}{a}$, $O_1B_1 = \frac{1}{b}$, $O_1C_1 = \frac{1}{c}$, and $O_1D_1 = \frac{1}{d}$. The equalities (1) and (2) imply that the opposite sides of $A_1B_1C_1D_1$ are equal in length, which means that $A_1B_1C_1D_1$ is a parallelogram. So $\frac{1}{a} = \frac{1}{c}$, and $\frac{1}{b} = \frac{1}{d}$, implying $a = c$ and $b = d$. This proves that $ABCD$ is a parallelogram.

II. *Solution by the proposer* (*slightly edited*).

If $ABCD$ is a trapezoid with $AB \parallel CD$, then

$$\frac{AB}{CD} \;=\; \frac{OK}{OM} \;=\; 1\,,$$

which means that $ABCD$ is a parallelogram. Hence, assume that $ABCD$ is not a trapezoid and set $P = AB \cap CD$, $Q = AD \cap BC$. Indeed, $P \neq Q$. The assumption on the given quadrilateral says exactly that PO bisects $\angle BPC$ and QO bisects $\angle AQB$. Thus

$$\frac{AP}{PC} \;=\; \frac{AO}{OC} \;=\; \frac{AQ}{QC}\,,$$

implying that P, O, and Q lie on the Apollonius circle with centre on the line AC. Similarly, since

$$\frac{BP}{PD} \;=\; \frac{BO}{OD} \;=\; \frac{BQ}{QD}\,,$$

P, O, and Q lie on the Apollonius circle with centre on the line BD. This implies that O is the circumcentre of $\triangle POQ$; that is, points P and Q coincide, a contradiction.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and D.J. SMEENK, Zaltbommel, the Netherlands. There were also five incorrect solutions submitted.

Most of the submitted solutions are similar to the proposer's solution.

2260. [1997: 301] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n}\, x\,.$$

Solution by Florian Herzig, student, Cambridge, UK; Gerry Leversha, St. Paul's School, London, England; Nick Lord, Tonbridge School, Tonbridge, Kent, England; and Panos E. Tsaoussoglou, Athens, Greece.

By the AM–GM Inequality applied to the $n+1$ positive numbers $x, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$, we have $\left(\frac{x+1}{n+1}\right)^{n+1} \geq \frac{x}{n^n}$, with equality if and only if $x = \frac{1}{n}$. This is clearly equivalent to the given inequality.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, Washington, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PAVLOS MARAGOUDAKIS, Hatzikiriakio, Pireas, Greece; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; JOHN VLACHAKIS, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; ROGER ZARNOWSKI, TREY SMITH, CHARLES DIMINNIE and GERALD ALLEN (jointly), Angelo State University, San Angelo, TX, USA; and the proposer. There were also three incomplete solutions.

The majority of solvers used a standard calculus approach to establish the given inequality. The only exceptions are the five listed in the solutions above plus Lambrou and Maragoudakis, who used Bernoulli's Inequality. Both Janous and Lambrou noted that the given inequality holds for all positive real n.

Janous also generalized the problem by showing that if $a > 0$, and $\alpha > \beta > 0$ are given real numbers, then the largest constant $C = C(a, \alpha, \beta)$ such that $(a + x)^\alpha \geq Cx^\beta$ holds for all $x > 0$ is given by $C = \left(\frac{a}{\alpha - \beta}\right)^{\alpha-\beta} \frac{\alpha^\alpha}{\beta^\beta}$. The given problem is the special case when $a = \beta = 1$ and $\alpha = n + 1$.

By applying the AM–GM Inequality to the $n + 1$ positive numbers: $1, \frac{x}{n}, \frac{x}{n}, \ldots, \frac{x}{n}$, Lord and the proposer obtained $(1 + x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n} x^n$, which is stronger than the proposed inequality for $x > 1$.

Zarnowski et al. *commented that when n is odd, the inequality is true for all real x, while if n is even, there is a number $x_n \leq -2$ such that the inequality holds for all $x \geq x_n$.*

2261. [1997: 301] *Proposed by Angel Dorito, Geld, Ontario.*

Assuming that the limit exists, find

$$\lim_{N\to\infty}\left(1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}}\right),$$

where every fraction in this expression has the form

$$\frac{a + \frac{b+\cdots}{c+\cdots}}{b + \frac{c+\cdots}{a+\cdots}}$$

for some cyclic permutation a, b, c of 1, 2, N.

[*Proposer's comment*: this problem was suggested by Problem 4 of Round 21 of the International Mathematical Talent Search, *Mathematics and Informatics Quarterly*, Vol. 6, No. 2, p. 113.]

Solution by Keith Ekblaw, Walla Walla, Washington, USA.

It will be shown that

$$1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}} \longrightarrow \frac{1+\sqrt{5}}{2} \qquad \text{as} \quad N \to \infty.$$

First, consider

$$J_N = N + \frac{1 + \frac{2+\cdots}{N+\cdots}}{2 + \frac{N+\cdots}{1+\cdots}}.$$

Note that

$$\frac{1 + \frac{2+\cdots}{N+\cdots}}{2 + \frac{N+\cdots}{1+\cdots}} > 0.$$

Thus $J_N > N$ and hence $J_N \to \infty$ as $N \to \infty$. Now let

$$K_N = 1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}} = 1 + \frac{2 + \frac{J_N}{K_N}}{J_N} = 1 + \frac{2}{J_N} + \frac{1}{K_N}.$$

Thus as $N \to \infty$ (and hence $J_N \to \infty$), $K_N \to 1 + 1/K_N$. Letting $K = \lim_{N\to\infty} K_N$, we have $K = 1 + 1/K$ or $K^2 - K - 1 = 0$ and so the required limit is

$$K = \frac{1+\sqrt{5}}{2}\,.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer.

Hess calculates that if N is replaced by 3.5, then the expression inside the limit (which is itself a limit, actually) is equal to 2. Readers may like to find other "nice" triples of numbers a, b, c so that the expression

$$a + \frac{b + \frac{c+\cdots}{a+\cdots}}{c + \frac{a+\cdots}{b+\cdots}}$$

is rational, say.

The proposer notes (as can be seen from the above proof) that the answer is still the golden ratio if the 2's in the given expression are replaced by any constant real number.

2262. [1997: 301] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Consider two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\angle A \geq 90°$ and $\angle A' \geq 90°$ and whose sides satisfy $a > b \geq c$ and $a' > b' \geq c'$. Denote the altitudes to sides a and a' by h_a and h'_a.

Prove that (a) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bb'} + \dfrac{1}{cc'}$, (b) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bc'} + \dfrac{1}{b'c}$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

(a) By the Cauchy-Schwarz inequality on the vectors $\left(\dfrac{1}{b}, \dfrac{1}{c}\right)$ and $\left(\dfrac{1}{b'}, \dfrac{1}{c'}\right)$, we have

$$\frac{1}{bb'} + \frac{1}{cc'} \;\leq\; \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{\frac{1}{2}} \left(\frac{1}{b'^2} + \frac{1}{c'^2}\right)^{\frac{1}{2}}.$$

Now $\dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{b^2+c^2}{b^2c^2} \leq \dfrac{a^2}{b^2c^2}$, since $\angle A \geq 90°$.

Also $\dfrac{a^2}{b^2c^2} \leq \dfrac{a^2}{b^2c^2\sin^2 A} = \dfrac{1}{h_a^2}$, since $\frac{1}{2}ah_a$ and $\frac{1}{2}bc\sin A$ are both formulae for the area of $\triangle ABC$. Similarly for $\triangle A'B'C'$.

Hence $\dfrac{1}{bb'} + \dfrac{1}{cc'} \leq \dfrac{1}{h_a h'_a}$, as required.

The proof of part (b) is the same, except that it starts with the vectors $\left(\frac{1}{b}, \frac{1}{c}\right)$ and $\left(\frac{1}{c'}, \frac{1}{b'}\right)$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; GEORGE TSAPAKIDIS, Agrinio, Greece; JOHN VLACHAKIS, Athens, Greece; and the proposer.

Most of the submitted solutions are similar to the one given above. Several solvers pointed out that the restrictions on the sides are unnecessary, and that equality in (a) *occurs if and only if* $\angle A = \angle A' = 90^\circ$ *and* $b/c = b'/c'$ *and in* (b) *if and only if* $\angle A = \angle A' = 90^\circ$ *and* $b/c = c'/b'$.

Janous proved more generally that for any real number $p \geq 1$

$$\frac{1}{(h_a h'_a)^p} \geq \frac{1}{(bb')^p} + \frac{1}{(cc')^p}, \quad \text{and} \quad \frac{1}{(h_a h'_a)^p} \geq \frac{1}{(bc')^p} + \frac{1}{(b'c)^p}.$$

2263. [1997: 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle, and the internal bisectors of $\angle B$, $\angle C$, meet AC, AB at D, E, respectively. Suppose that $\angle BDE = 30^\circ$. Characterize $\triangle ABC$.

Solution by the proposer.

Let F be the reflection of E across BD. Since $\angle EBD = \angle CBD$, it follows that F lies on BC. So $\angle BDF = \angle BDE = 30^\circ$, and $DF = DE$. Then $\triangle DEF$ is equilateral, so $EF = ED$ and $\angle FED = 60^\circ$. By the Law of Sines for $\triangle EFC$ and $\triangle EDC$ we obtain that

$$\frac{EC}{\sin\angle EFC} = \frac{EF}{\sin\angle ECF} = \frac{ED}{\sin\angle ECD} = \frac{EC}{\sin\angle EDC},$$

which gives $\sin\angle EFC = \sin\angle EDC$. It follows that we have either that $\angle EFC = \angle EDC$, or that $\angle EFC + \angle EDC = 180^\circ$.

Case 1. $\angle EFC = \angle EDC$. Then

$$\angle FEC = \angle DEC = \tfrac{1}{2}\angle FED = 30^\circ.$$

Let I be the intersection of BD and CE. Then

$$\angle DIC = \angle IED + \angle IDE = 30^\circ + 30^\circ = 60^\circ.$$

Since $\angle DIC = 90^\circ - \frac{1}{2}\angle A$, we obtain $90^\circ - \frac{1}{2}\angle A = 60^\circ$, so $\angle A = 60^\circ$.

Case 2. $\angle EFC + \angle EDC = 180^\circ$. Then
$\angle FED + \angle FCD = 180^\circ$, so that $60^\circ + \angle FCD = 180^\circ$.
Thus $\angle FCD = 120^\circ$; that is, $\angle ACB = 120^\circ$.

Therefore, ABC is a triangle with either $\angle A = 60°$ or $\angle C = 120°$.

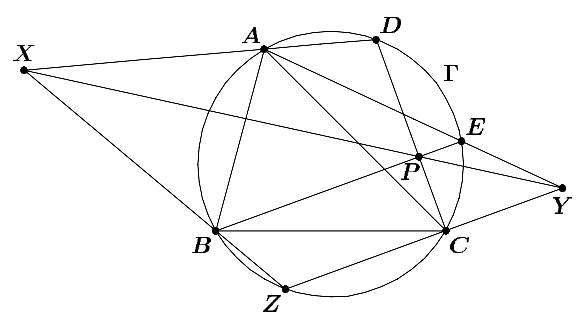
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; D.J. SMEENK, Zaltbommel, the Netherlands. There were also two incomplete solutions submitted.

Herzig and Lambrou have also shown that the characteristic condition is sufficient, that is, $\angle A = 60°$ or $\angle C = 120°$ implies $\angle BDE = 30°$.

2265. [1997: 364] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given triangle ABC, let ABX and ACY be two variable triangles constructed outwardly on sides AB and AC of $\triangle ABC$, such that the angles $\angle XAB$ and $\angle YAC$ are fixed and $\angle XBA + \angle YCA = 180°$. Prove that all the lines XY pass through a common point.

Solution by Toshio Seimiya, Kawasaki, Japan.



We denote the circumcircle of $\triangle ABC$ by Γ. Let BX and CY meet at Z. Since $\angle XBA + \angle YCA = 180°$, we get $\angle XBA = 180° - \angle YCA = \angle ACZ$, so that A, B, Z, C are concyclic, that is, Z lies on Γ. Let D, E be the second intersections of AX, AY respectively with Γ. Since AX and AY are fixed lines, D and E are fixed points. Let P be the intersection of BE and CD. Since hexagon $ADCZBE$ is inscribed in Γ, by Pascal's Theorem the intersections of AD and BZ, of DC and BE, and of CZ and EA are collinear. Therefore variable line XY always passes through the fixed point P. [*Editorial note*: if the diagram differs from the one shown, for example if Z lies between X and B, the proof still works with minor changes.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, *Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. One incorrect solution and one comment were sent in.*

Seimiya and the proposer had similar solutions.

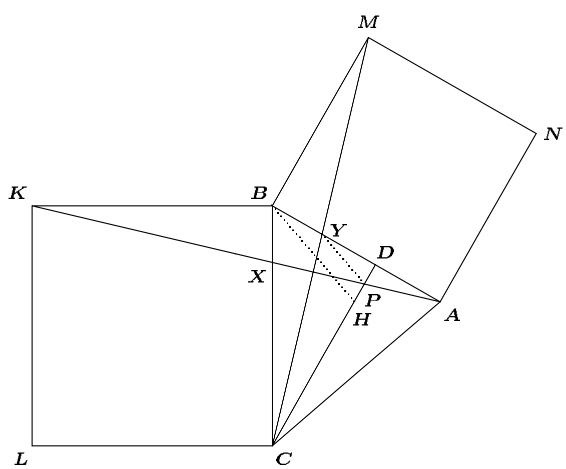
Herzig notes that, if the fixed angles are chosen to be AXB and AYC instead, then the lines XY still pass through a fixed point. Readers may like to show this themselves.

2266. [1997: 364] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

$BCLK$ is the square constructed outwardly on side BC of an acute triangle ABC. Let CD be the altitude of $\triangle ABC$ (with D on AB), and let H be the orthocentre of $\triangle ABC$. If the lines AK and CD meet at P, show that

$$\frac{HP}{PD} = \frac{AB}{CD}.$$

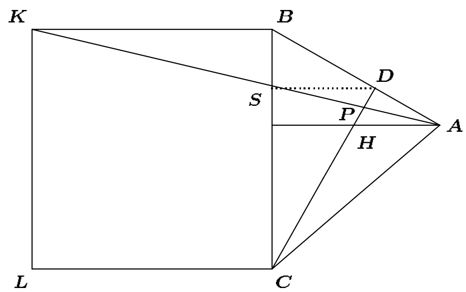
I. *Solution by Florian Herzig, student, Cambridge, UK.*



Construct a square $ABMN$ outwardly on $\triangle ABC$. Let X and Y be the points of intersection of AK, BC and AB, CM respectively. The rotation through a right angle about B maps $\triangle MBC$ onto $\triangle ABK$. Hence AK and CM are perpendicular and it follows that AP and CD are altitudes in $\triangle AYC$. Therefore P is the orthocentre in that triangle, and as a consequence $YP \perp AC$ or $YP \parallel BH$. Thus

$$\frac{HP}{PD} = \frac{BY}{YD} = \frac{BM}{CD} = \frac{AB}{CD}.$$

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*



Let S be a point on AK such that $DS \perp BC$ and so $DS \parallel AH \parallel BK$. Since $AH \parallel DS$ we get

$$\frac{HP}{PD} = \frac{AH}{DS}. \tag{1}$$

Since $DS \parallel BK$ and $BK = BC$, we have

$$\frac{AD}{AB} = \frac{DS}{BK} = \frac{DS}{BC}. \tag{2}$$

Since $AH \perp BC$ and $CD \perp AB$ we get $\angle HAD = \angle BCD$. Moreover we have $\angle HDA = \angle BDC (= 90^\circ)$, so that $\triangle HAD \sim \triangle BCD$. Thus

$$\frac{AH}{BC} = \frac{AD}{CD}. \tag{3}$$

From (2) and (3) we have

$$\frac{AH}{DS} = \frac{AH}{BC} \cdot \frac{BC}{DS} = \frac{AD}{CD} \cdot \frac{AB}{AD} = \frac{AB}{CD},$$

so that we obtain from (1) that $\dfrac{HP}{PD} = \dfrac{AB}{CD}$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; JOHN VLACHAKIS, Athens, Greece; and the proposer.

The proposer's solution was the same as Herzig's. Most other solvers used either similar triangles (as in II*), trigonometry, or coordinates.*

2268. [1997: 364] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let x, y be real. Find all solutions of the equation

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}} \;=\; \sqrt{xy}+\frac{x+y}{2}.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $A=\sqrt{\dfrac{x^2+y^2}{2}}$ and $B=\sqrt{xy}$. Then

$$2(A^2+B^2)=(x+y)^2 \qquad \text{and} \qquad 2(A^2-B^2)=(x-y)^2$$

and the given equation yields

$$\begin{aligned} A-B &= \frac{x+y}{2}-\frac{2xy}{x+y} \\ \text{or} \quad \frac{2(A^2-B^2)}{A+B} &= \frac{(x-y)^2}{x+y} \\ \frac{(x-y)^2}{A+B} &= \frac{(x-y)^2}{x+y}. \end{aligned}$$

Therefore, we have either $(x-y)^2=0$ (which implies that $x=y$) or $A+B=x+y$. Let us consider $A+B=x+y$:

$$\begin{aligned} A+B=x+y \quad &\iff \quad (A+B)^2=2(A^2+B^2) \\ &\iff \quad (A-B)^2=0 \quad \iff \quad A=B \\ &\iff \quad \sqrt{\frac{x^2+y^2}{2}}=\sqrt{xy} \\ &\iff \quad (x-y)^2=0 \quad \iff \quad x=y \end{aligned}$$

In conclusion, all solutions have $x=y\neq 0$, because $x+y\neq 0$.

Also solved by HAYO AHLBURG, Benidorm, Spain; PAUL BRACKEN, CRM, Université de Montréal; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark (2 solutions); CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; ROGER

ZARNOWSKI, Angelo State University, San Angelo, Texas, USA; and the proposer. There were 14 incorrect solutions submitted, 11 of which simply did NOT exclude the origin from the solution set.

2270. [1997: 365] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given $\triangle ABC$ with sides a, b, c, a circle, centre P and radius ρ intersects sides BC, CA, AB in A_1 and A_2, B_1 and B_2, C_1 and C_2 respectively, so that

$$\frac{\overline{A_1A_2}}{a} = \frac{\overline{B_1B_2}}{b} = \frac{\overline{C_1C_2}}{c} = \lambda \geq 0.$$

Determine the locus of P.

Solution by the proposer.

[Assume that no two sides of the triangle are equal.] The distance from

$$P \text{ to } BC \text{ is } \quad x = \sqrt{\rho^2 - \frac{\lambda^2 a^2}{4}},$$

$$P \text{ to } CA \text{ is } \quad y = \sqrt{\rho^2 - \frac{\lambda^2 b^2}{4}},$$

$$P \text{ to } AB \text{ is } \quad z = \sqrt{\rho^2 - \frac{\lambda^2 c^2}{4}}.$$

It follows that $\dfrac{x^2 - y^2}{y^2 - z^2} = \dfrac{b^2 - a^2}{c^2 - b^2}$, or

$$x^2(c^2 - b^2) + y^2(a^2 - c^2) + z^2(b^2 - a^2) = 0. \qquad (1)$$

Considering x, y, z to be the triangular coordinates of P with respect to $\triangle ABC$, we conclude that (1) represents a conic K. Note that K passes through the incentre $I(1,1,1)$ and the excentres $I_a(-1,1,1)$, $I_b(1,-1,1)$, and $I_c(1,1,-1)$, [and also the circumcentre $O(\cos A, \cos B, \cos C)$]. So K is the conic through O of the pencil determined by I, I_a, I_b, I_c. Since the degenerate conics of the pencil are degenerate orthogonal hyperbolas (that is, pairs of perpendicular lines), K must be an orthogonal hyperbola.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

Neither solver mentioned the obvious special cases:

The locus is a pair of perpendicular lines when $\triangle ABC$ is isosceles, and just the four points I, I_a, I_b, I_c when equilateral.

Bradley points out that it is clear from the statement of the problem (with no need for coordinates) that the locus includes I, I_a, I_b, I_c (when $\lambda = 0$) and O (when $\lambda = 1$).

2271. [1997: 365] *Proposed by F.R. Baudert, Waterkloof Ridge, South Africa.*

A municipality charges householders per month for electricity used according to the following scale:

first 400 units — 4.5¢ per unit;

next 1100 units — 6.1¢ per unit;

thereafter — 5.9¢ per unit.

If E is the total amount owing (in dollars) for n units of electricity used, find a closed form expression, $E(n)$.

Solution by Michael Lambrou, University of Crete, Greece.

We may view the charges as consisting of

(i) 4.5¢ per unit and, additionally,

(ii) a surcharge of $6.1 - 4.5 = 1.6$¢ per unit, but with

(iii) a refund of $6.1 - 5.9 = 0.2$¢ per unit for units consumed in excess of $400 + 1100 = 1500$.

So (in dollars) the amount owing for n units is:

$$\frac{45}{1000}n + \frac{16}{1000}\max\{0, n - 400\} - \frac{2}{1000}\max\{0, n - 1500\}.$$

Now writing $\max\{a, b\} = \frac{1}{2}\left(|a - b| + a + b\right)$, this simplifies to:

$$\frac{1}{1000}\left\{52n + 8|n - 400| - |n - 1500| - 1700\right\}.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer. There were two incorrect solutions submitted. Some solvers used unit step functions instead of absolute value to express the answer.

2272*. [1997: 365] *Proposer unknown (please identify yourself!)*

Write $r \precsim s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n $(n \geq 2)$, the least positive integer satisfying

$$\frac{k}{n} \precsim \frac{k}{n-1} \precsim \frac{k}{n-2} \precsim \cdots \precsim \frac{k}{2} \precsim k.$$

Solution by Florian Herzig, student, Cambridge, UK (modified by the editor).

Let k_n denote the least positive integer k satisfying

$$\frac{k}{n} \precsim \frac{k}{n-1} \precsim \frac{k}{n-2} \precsim \cdots \preceq \frac{k}{2} \precsim k \qquad (1)$$

We claim that $k_n = \left\lfloor \left(\frac{n+x_n}{2}\right)^2 + 1 \right\rfloor$, where $x_n = \lfloor n+1-2\sqrt{n-1} \rfloor$. We first show that k_n satisfies (1). To this end, let $u = \lfloor \sqrt{k_n} \rfloor$. Then for all integers $a = 1, 2, \cdots, u-1$ we have, from $a \le \sqrt{k_n} - 1$ that $a(a+1) \le \sqrt{k_n}(\sqrt{k_n} - 1) < k_n$. Hence $\frac{k_n}{a} - \frac{k_n}{a+1} > 1$, which implies $\frac{k_n}{a+1} \precsim \frac{k_n}{a}$. Therefore we have

$$\frac{k_n}{u} \precsim \frac{k_n}{u-1} \precsim \cdots \precsim \frac{k_n}{2} \preceq k_n. \qquad (2)$$

Next we show that for all integers $a = u+1, u+2, \cdots, n$

$$n + x_n - a < \frac{k_n}{a} < n + x_n - a + 1. \qquad (3)$$

In fact, the left inequality in (3) holds for all $a = 1, 2, \cdots, n$. To see this, note that from $k_n > \left(\frac{n+x_n}{2}\right)^2$ we get $a^2 - ax_n - an + k_n = \left(a - \frac{n+x_n}{2}\right)^2 + k_n - \left(\frac{n+x_n}{2}\right)^2 > 0$, and thus $n + x_n - a < \frac{k_n}{a}$.

On the other hand, note that the right inequality in (3) is equivalent to

$$\left(a - \frac{n+x_n+1}{2}\right)^2 < \left(\frac{n+x_n+1}{2}\right)^2 - k_n. \qquad (4)$$

Since $k_n > \left(\frac{n+x_n}{2}\right)^2$, we have

$$a \ge u + 1 = \left\lfloor \sqrt{k_n} \right\rfloor + 1 \ge \left\lfloor \frac{n+x_n}{2} \right\rfloor + 1 \ge \frac{n+x_n+1}{2}.$$

[*Ed: The last inequality holds since $n + x_n$ is an integer.*]

Hence it suffices to establish (4) for $a = n$.

Substituting $a = n$ into the right inequality of (3), we need to show that $k_n < nx_n + n$. From $n - 2\sqrt{n-1} < x_n < n$ we get $(n - x_n)^2 < 4(n-1)$ or $(n+x_n)^2 + 4 < 4(nx_n + n)$. Hence $k_n \le \left(\frac{n+x_n}{2}\right)^2 + 1 < nx_n + n$. Therefore (3) holds, and by setting $a = u+1, u+2, \cdots, n$ we get

$$\frac{k_n}{n} < x_n + 1 < \frac{k_n}{n-1} < \cdots < x_n + n - u - 1 < \frac{k_n}{u+1} < x_n + n - u < \frac{k_n}{u}.$$

Hence

$$\frac{k_n}{n} \precsim \frac{k_n}{n-1} \cdots \precsim \frac{k_n}{u+1} \precsim \frac{k_n}{u}. \tag{5}$$

From (2) and (5) we conclude that k_n satisfies (1).

Now we show that if k is any integer satisfying (1), then $k \geq k_n$. To this end, let $x = \left\lfloor \frac{k}{n} \right\rfloor$.We first show that $x \geq x_n$. Since there exists an integer z such that $\frac{k}{n} < z < \frac{k}{n-1}$ and $x = \left\lfloor \frac{k}{n} \right\rfloor < z$, we have $z - x \geq 1$ and hence $\frac{k}{n-1} > x + 1$. Similarly, $\frac{k}{n-2} > x + 2$, $\frac{k}{n-3} > x + 3, \cdots$, $\frac{k}{1} > x + n - 1$. That is, for all $a = 1, 2, \cdots, n$ we have $k > (n-a)(x+a)$

$$\text{or} \quad k \geq (n-a)(x+a) + 1 = -a^2 + a(n-x) + nx + 1. \tag{6}$$

Hence

$$k \geq \left(\frac{n-x}{2}\right)^2 + nx + 1 - \left(a - \frac{n-x}{2}\right)^2. \tag{7}$$

Note that $x \geq 1$. [*Ed: If* $k \leq n-1$, *then* $0 < \frac{k}{n} < \frac{k}{n-1} \leq 1$, *contradicting* $\frac{k}{n} \precsim \frac{k}{n-1}$. *Hence* $k \geq n$].

On the other hand, it is clear that $x_n \leq n-1$. Suppose, contrary to what we claim, that $x < x_n$. Then we have $1 \leq x < x_n \leq n-1$ and so $2 \leq n - x < n$ or $1 \leq \frac{n-x}{2} < \frac{n}{2}$. (Here we must assume that $n \geq 3$. The case when $n = 2$ can be treated separately, since it is easy to verify that $k_2 = 3$.) Hence we may let $a = \left\lfloor \frac{n-x}{2} \right\rfloor$ in (6) and (7) and obtain

$$k \geq \left(\frac{n-x}{2}\right)^2 + nx + 1 - \left(\left\lfloor \frac{n-x}{2} \right\rfloor - \left(\frac{n-x}{2}\right)\right)^2. \tag{8}$$

Since the right side of (7) is an integer and since the last squared term in (8) is either 0 or $\frac{1}{4}$ we get

$$k \geq \left\lfloor \left(\frac{n-x}{2}\right)^2 + nx + 1 \right\rfloor = \left\lfloor \left(\frac{n+x}{2}\right)^2 + 1 \right\rfloor. \tag{9}$$

Thus $x = \left\lfloor \frac{k}{n} \right\rfloor \geq \left\lfloor \frac{1}{n} \left\lfloor \left(\frac{n+x}{2}\right)^2 + 1 \right\rfloor \right\rfloor = \left\lfloor \frac{1}{n} \left(\frac{n+x}{2}\right)^2 + \frac{1}{n} \right\rfloor$.

[*Ed: It is known and easy to show that* $\left\lfloor \frac{\lfloor z \rfloor}{n} \right\rfloor = \left\lfloor \frac{z}{n} \right\rfloor$ *for all real numbers* z *and positive integers* n.]

Hence $0 \geq \left\lfloor \frac{1}{n}\left(\frac{n+x}{2}\right)^2 + \frac{1}{n} \right\rfloor - x = \left\lfloor \frac{1}{n}\left(\frac{n-x}{2}\right)^2 + \frac{1}{n} \right\rfloor$, which implies that $\frac{1}{n}\left(\left(\frac{n-x}{2}\right)^2 + 1\right) \leq 1$ or $\left(\frac{n-x}{2}\right)^2 \leq n-1$.

Thus $n - x < 2\sqrt{n-1}$ or $x > n - 2\sqrt{n-1}$, from which we get $x \geq \lfloor n+1-2\sqrt{n-1} \rfloor = x_n$, a contradiction. Hence $x \geq x_n$. Therefore we may replace x by x_n in (6), (7), and (9) and conclude that $k \geq \left\lfloor \left(\frac{n+x_n}{2}\right)^2 + 1 \right\rfloor = k_n$. This completes the proof.

Also solved by PETER TINGLEY, student, University of Waterloo, Waterloo, Ontario. There was one incorrect solution. Tingley gave the answer

$$k_n = n\lfloor n - 2\sqrt{n-1} + 1\rfloor + \left\lfloor \left(\frac{n - \lfloor n - 2\sqrt{n-1} + 1\rfloor}{2}\right)^2 + 1 \right\rfloor$$

which is readily seen to be the same as the one obtained by Herzig. The proposer had conjectured that

$$k_n = \begin{cases} 1 + (n-m)^2 & \text{if } m^2 \leq n-2 \\ 1 + (n-m)^2 + (n-m) & \text{otherwise} \end{cases}$$

where $m = \left\lfloor \frac{1+\sqrt{4n-7}}{2} \right\rfloor$ *and had verified it for* $2 \leq n \leq 600$ *using a computer. In a private communication Tingley has actually proved that this conjectured formula is equivalent to the answer given by Herzig and himself. Interested readers may find the proof of this fact quite challenging.*

2273. [1997: 366] *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Consider the sequence of positive integers: $\{1, 12, 123, 1\,234, 12\,345, \ldots\}$, where the next term is constructed by lengthening the previous term at its right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying" occurring as in addition: thus the ninth and tenth terms of the sequence are $123\,456\,789$ and $1\,234\,567\,900$ respectively.

Determine which terms of the sequence are divisible by 7.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

The sequence $\{a_n\}$ under consideration satisfies the recurrence

$$a_1 = 1 \quad \text{and} \quad a_n = 10a_{n-1} + n \quad \text{for } n \geq 2.$$

A simple induction argument shows that

$$81a_n = 10^{n+1} - 9n - 10, \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. Applying the Euclidean Algorithm twice, we see that there exist non-negative integers j, k, r such that $0 \le k \le 6$, $0 \le r \le 5$, and $n = 42j + 6k + r$. Since $3^{42j} \equiv 3^{6k} \equiv 1 \pmod 7$ by Fermat's Little Theorem, it follows that

$$4a_n \equiv 3^{r+1} + 2k - 2r - 3 \pmod 7.$$

The following table gives the remainder when the expression on the right hand side of the above congruence is divided by 7:

$r \backslash k$	0	1	2	3	4	5	6
0	0	2	4	6	1	3	5
1	4	6	1	3	5	0	2
2	6	1	3	5	0	2	4
3	2	4	6	1	3	5	0
4	1	3	5	0	2	4	6
5	2	4	6	1	3	5	0

Inspecting this table and using the above congruence, we see that a_n is divisible by 7 if and only if $n \equiv 0, 22, 26, 31, 39,$ or $41 \pmod{42}$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; J.A. MCCALLUM, Medicine Hat, Alberta; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; TREY SMITH, GERALD ALLEN, NOEL EVANS, CHARLES DIMINNIE, AND ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer. There was one incorrect solution submitted.

2274. [1997: 366] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

A. Let m be a non-negative integer. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)$.

B. Let $m \in \{1, 2, 3, 4\}$. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^2$.

C⋆. Let m and α_j ($j = 0, 1, \ldots, m$) be non-negative integers. Prove or disprove that $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^{\alpha_j}$ is divisible by $\displaystyle\prod_{j=0}^{m+1}(n+j)$.

I. *Solution by Florian Herzig, student, Cambridge, UK.*

A. Note that

$$\begin{aligned}\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j) &= \sum_{k=1}^{n}\frac{(m+k)!}{(k-1)!} = (m+1)!\sum_{k=1}^{n}\binom{m+k}{k-1}\\ &= (m+1)!\left[\binom{m+1}{0}+\binom{m+2}{1}+\binom{m+3}{2}+\cdots+\binom{m+n}{n-1}\right].\end{aligned}$$

This combinatorial sum [inside the square brackets] is well-known and can be evaluated as follows. Note that the first two terms add to $\binom{m+3}{1}$ which in turn adds with the third term to $\binom{m+4}{2}$, and so on until we obtain the desired closed form $\binom{m+n+1}{n-1}$ in the end. Thus

$$\begin{aligned}\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j) &= (m+1)!\binom{m+n+1}{n-1}\\ &= \frac{(m+n+1)!}{(m+2)(n-1)!} = \frac{n(n+1)\ldots(n+m+1)}{m+2}.\end{aligned}$$

B. The expressions reduce to sums of the form $\sum_{k=1}^{n}k^m$. With the help of a calculator I got

$$\binom{n+2}{3}\frac{2(3n^2+6n+1)}{5}\quad\text{for } m=1,$$

$$\binom{n+3}{4}\frac{12(2n+3)(5n^2+15n+1)}{35}\quad\text{for } m=2,$$

$$\binom{n+4}{5}\frac{8(35n^4+280n^3+685n^2+500n+12)}{21}\quad\text{for } m=3,$$

$$\binom{n+5}{6}\frac{40(126n^5+1575n^4+6860n^3+12075n^2+7024n+60)}{77}\quad\text{for } m=4.$$

C. I assume that "is divisible by" means *polynomial* division.

Clearly the α_j should be *positive* integers, and in this case I will prove that the claim is true. Define

$$P(n) = \sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^{\alpha_j},$$

a polynomial of degree $(\sum_{j=1}^{n} \alpha_j) + 1$ (since each $\sum_{k=1}^{n} k^l$ is a polynomial of degree $l + 1$). Then

$$P(n) = \sum_{k=-m}^{n} \prod_{j=0}^{m} (k+j)^{\alpha_j} = \sum_{k=1}^{m+n+1} \prod_{j=0}^{m} (k-m-1+j)^{\alpha_j},$$

as in the first of these sums all the terms for $k \le 0$ vanish. Hence, for all integers a such that $-m \le a \le 0$ we get

$$P(a) = \sum_{k=1}^{m+a+1} (k-m-1)^{\alpha_0}(k-m)^{\alpha_1} \ldots (k-1)^{\alpha_m} = 0,$$

since each term is zero. [Thus $n - a$ must be a factor of $P(n)$ for each such a, so $P(n)$ must be divisible by each of $n, n+1, \ldots, n+m$. — *Ed.*] For $n = -m - 1$ the above sum is empty and so $P(-m-1) = 0$ as well (to avoid empty sums one can instead write $P(n)$ as $\sum_{k=0}^{m+n+1} p(k) - p(0)$ where $p(k)$ is the above product). Therefore $P(n)$ is divisible by $\prod_{j=0}^{m+1}(n+j)$ as claimed.

II. *Solution to part* A *by Michael Lambrou, University of Crete, Greece.*

A. From the identity

$$\prod_{j=0}^{m} (k+j) = \frac{1}{m+2}\left[\prod_{j=0}^{m+1} (k+j) - \prod_{j=0}^{m+1} (k-1+j)\right]$$

(easily verified by considering the common factor $\prod_{j=0}^{m}(k+j)$ of the two products on the right), we obtain telescopically

$$\sum_{k=1}^{n} \prod_{j=0}^{m} (k+j) = \frac{1}{m+2}\left[\prod_{j=0}^{m+1} (n+j) - \prod_{j=0}^{m+1} j\right] = \frac{1}{m+2}\prod_{j=0}^{m+1} (n+j).$$

[*Editorial note*: Lambrou also solved parts B and C.]

All three parts also solved by G. P. HENDERSON, Garden Hill, Ontario. Parts A *and* B *only solved by THEODORE N. CHRONIS, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

As Herzig mentions above, part A *at least is a fairly familiar result. (For example, see formula 2.50, page 50 of* Concrete Mathematics *by Graham, Knuth and Patashnik.) In fact it was proposed for publication in* ***CRUX*** *in 1991 by an Edmonton high school student, Jason Colwell, but was not accepted by the then editor.*

Chronis notes that, in the solution for part B, *when* $m = 4$, *the fifth degree polynomial has* $2n + 5$ *as a factor.*

2275. [1997: 366] *Proposed by M. Perisastry, Vizianagaram, Andhra Pradesh, India.*

Let $b > 0$ and $b^a \geq ba$ for all $a > 0$. Prove that $b = e$.

I. *Solution by Gerald Allen, Charles Diminnie, Trey Smith and Roger Zarnowski (jointly), Angelo State University, San Angelo, TX, USA; Russell Euler and Jawad Sadek (jointly), NW Missouri State University, Maryville, Missouri, USA; Michael Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland; Reza Shahidi, student, University of Waterloo, Waterloo, Ontario; George Tsapakidis, Agrinio, Greece; and John Vlachakis, Athens, Greece.*

Let $f(x) = b^x - bx$ for $x > 0$. Since $f(x) \geq 0$ for all $x > 0$ and $f(1) = 0$, it follows that f has a relative (as well as an absolute) minimum at $x = 1$.

Since $f'(1)$ exists, we have $f'(1) = 0$; that is, $b \ln b - b = 0$. Since $b > 0$, we get $\ln b = 1$ or $b = e$.

II. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece; Florian Herzig, student, Cambridge, UK; Michael Lambrou, University of Crete, Crete, Greece; Gerry Leversha, St. Paul's School, London, England; Vedula N. Murty, Visakhapatnam, India; Heinz-Jürgen Seiffert, Berlin, Germany; and David R. Stone, Georgia Southern University, Statesboro, Georgia, USA.*

The given inequality is equivalent to $b^{a-1} \geq a$ for all $a > 0$. Letting $a = 1 + \dfrac{1}{n}$ where $n \in \mathbb{N}$, we get $b^{\frac{1}{n}} \geq 1 + \dfrac{1}{n}$, or $b \geq \left(1 + \dfrac{1}{n}\right)^n$. Hence $b \geq \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

On the other hand, letting $a = \left(1 + \dfrac{1}{n}\right)^{-1} = \dfrac{n}{n+1}$, where $n \in \mathbb{N}$, we get from $b^{1-a} \leq \dfrac{1}{a}$, that $b^{\frac{1}{n+1}} \leq 1 + \dfrac{1}{n}$, or $b \leq \left(1 + \dfrac{1}{n}\right)^{n+1}$. Hence $b \leq \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$. Therefore, $b = e$.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MA, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; LUZ M. DeALBA, Drake University, Des Moines, IA, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the *proposer. There was one incorrect solution.*

Although the problem did not ask to show that the condition $b = e$ is both necessary and sufficient, a few solvers did provide a proof of the simple fact that $e^x \geq ex$ for all $x > 0$.

2277. [1997: 431] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

For $n \geq 1$, define

$$u_n = \left[\frac{1}{(1,n)}, \frac{2}{(2,n)}, \dots, \frac{n-1}{(n-1,n)}, \frac{n}{(n,n)}\right],$$

where the square brackets [ ] and the parentheses ( ) denote the **least common multiple** and **greatest common divisor** respectively.

For what values of n does the identity $u_n = (n-1)u_{n-1}$ hold?

Solution by Florian Herzig, student, Cambridge, UK.

We first introduce some notation:

for any prime p, let $(n)_p = \max\{\alpha | p^\alpha \leq n\}$, $[n]_p = \max\{\alpha | p^\alpha \text{divides } n\}$. Thus $p^{[n]_p} || n$. For each $k = 1, 2, \cdots, n$ we determine $a = a(k)$ such that $p^a || k$. If $a \leq [n]_p$, then since $p^a | n$, we have $p^a || (k,n)$ and so $p \not| \dfrac{k}{(k,n)}$. If $a > [n]_p$, then since $p^{[n]_p} | k$ we have $p^{[n]_p} || (k,n)$ and so $p^{a-[n]_p} || \dfrac{k}{(k,n)}$. Thus the highest power of p in any $\dfrac{k}{(k,n)}$ arises when $a = (n)_p$. This shows that $u_n = \prod_p p^{(n)_p - [n]_p}$, where the product is over all primes. Note that $n - 1 = \prod_p p^{[n-1]_p}$ and hence $u_n = (n-1)u_{n-1}$ is equivalent to

$$(n)_p - [n]_p = (n-1)_p \text{ for all primes } p \qquad (1)$$

We distinguish two cases:

Case (i) Suppose n is a prime power, say $n = q^b$ where q is a prime and $b > 0$. For $p \neq q$, (1) is satisfied since $(n)_p = (n-1)_p$ and $[n]_p = 0$. For $p = q$ we have $n = p^b$ and so $(n)_p = [n]_p = b$ and $(n-1)_p = b - 1$. Hence (1) holds if and only if $b - 1 = 0$; that is, $b = 1$.

Case (ii) If n is not a prime power, then $(n)_p = (n-1)_p$ for all primes p. Hence (i) holds if and only if $[n]_p = 0$ for all primes p, and so $n = 1$. [*Ed: Clearly $n = 1$ is not a solution, since u_0 is undefined.*]

Therefore $u_n = (n-1)u_{n-1}$ if and only if n is a prime.

Also solved by ED BARBEAU, University of Toronto, Toronto; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

From the proof given above, it is not difficult to see that, in fact, we have $u_n = [1, 2, \cdots, n]/n$. *This was explicitly pointed out by Konečný, Lambrou, and the proposer, but only Lambrou and the proposer actually gave a proof.*

2278. [1997: 431] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Determine the value of a_n, which is the number of ordered n-tuples $(k_2, k_3, \ldots, k_n, k_{n+1})$ of non-negative integers such that

$$2k_2 + 3k_3 + \ldots + nk_n + (n+1)k_{n+1} = n + 1.$$

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

We show that $a_n = p(n+1) - p(n)$ for $n \geq 1$ where $p(m)$ denotes the number of partitions of m into positive integral parts. Our argument is based on the well-known observation that to a partition of m where l_k k's appear ($k = 1, 2, \ldots, m$), so that

$$1l_1 + 2l_2 + \cdots + ml_m = m, \tag{1}$$

corresponds the ordered m-tuple $(l_1, l_2, \ldots, l_m)$. Conversely, to any given ordered m-tuple $(l_1, l_2, \ldots, l_m)$ of positive integers satisfying (1), there corresponds a partition of m.

For fixed $n \geq 1$ consider the partitions of $n+1$ as above. They are of two types:

(a) those for which the number 1 is absent in the decomposition; or

(b) those for which the number 1 appears at least once.

The number of partitions of type (a) is clearly a_n. Moreover, for each partition in case (b), if we delete one 1, we get a partition of n. Conversely, every partition of n with an extra 1 added on gives a partition of $n+1$ of type (b). Clearly then $p(n+1) = a_n + p(n)$, as required.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We deal more generally with the equation:

$$k_1 + 2k_2 + \cdots + (j-1)k_{j-1} + (j+1)k_{j+1} + \cdots + (n+1)k_{n+1} = n+1$$

where $j \in \{1, 2, \ldots, n+1\}$ is fixed and determine the number $a_n(j)$ of its non-negative solutions in $\mathbb{Z}^n$.

For this we recall that (since Euler's days) such problems are dealt with best by generating functions, namely:

$$\begin{aligned} E(x) &= (1 + x^1 + x^{2\cdot 1} + \dots) \cdot (1 + x^2 + x^{2\cdot 2} + \dots) \cdot \dots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \dots = \sum_{k=0}^{\infty} p(k) x^k \end{aligned}$$

where $p(k)$ denotes the number of partitions of k; that is, the number of unordered representations of k as $k = s_1 + s_2 + \dots + s_e$ with s_j a positive integer for $j = 1, 2, \dots, e$, or equivalently $k = 1n_1 + 2n_2 + \dots + kn_k$, where $n_j \geq 0$ is the number of appearances of summand j. Therefore, all partitions with summand j forbidden are obtained via

$$(1-x^j)E(x) = (1-x^j) \sum_{k=0}^{\infty} p(k) x^k = \sum_{k=0}^{j-1} p(k) x^k + \sum_{k=j}^{\infty} (p(k) - p(k-j)) x^k .$$

Hence the desired amount $a_n(j)$ equals:

$$a_n(j) = \begin{cases} p(n+1), & \text{if } n+1 \leq j-1 \\ p(n+1) - p(n+1-j), & \text{if } n+1 \geq j \end{cases}$$

Also solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany. There was one incorrect solution submitted.

Janous remarks how his ideas above can be extended to include the case where the excluded summand can be a subset of the values from 1 *to* $n+1$*.*