MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

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Shreds and Slices

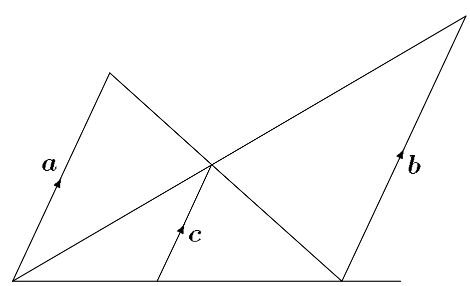
An Algebraic Relation with a Geometric Twist

Cyrus Hsia

Consider the following algebraic relationship between the positive real numbers a, b, and c:

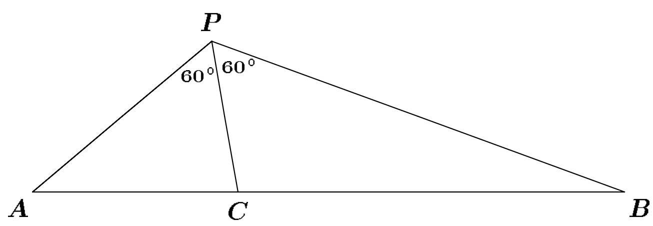
$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}.$$

If we consider line segments with lengths of a, b, and c, then they are related to each other as shown in the following diagram.



The diagram shows the three line segments parallel to each other and emanating from a common line. This figure and relationship between the line segments appear a lot. The reader is encouraged to prove this.

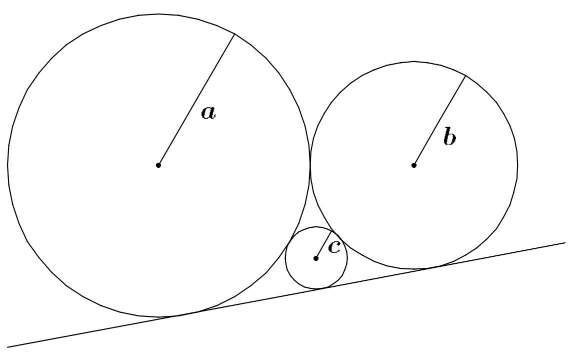
As a corollary of this fact, here is another geometric example. Let P, A, B, and C be points in the plane such that $\angle APC = \angle CPB = 60°$ and A, C, and B are collinear. Show that $1/PC = 1/PA + 1/PB$.



Another algebraic relation between three positive numbers with an interesting geometric interpretation is the following:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

The reader is encouraged to find a geometric interpretation for the above relation before looking at the diagram below. Use a, b, and c as the length of three line segments, and determine a geometric figure that relates the three.



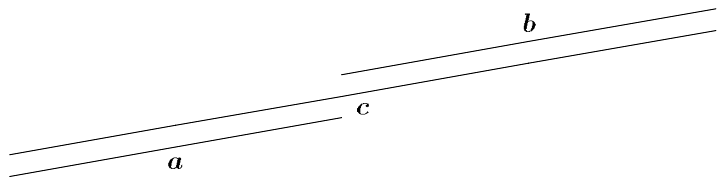
It turns out that if a, b, and c are considered to be the lengths of the radii of three circles, then the circles may all be tangent to a common line and to each other as shown. Again readers are encouraged to prove this themselves.

Now what about a generalization? Consider the following algebraic relation between positive reals a, b, c, and a real number x:

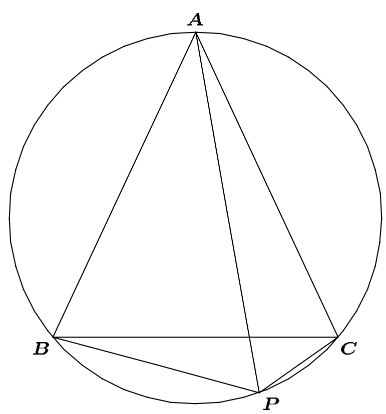
$$c^x = a^x + b^x.$$

The first case then corresponds to the value $x = -1$ and the second case to $x = -\frac{1}{2}$.

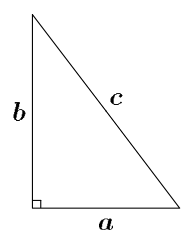
The case $x = 1$ is trivial, as we could interpret it geometrically as a line segment of length c is made up of the sum of its parts of lengths a and b.



If we wanted to get fancy, we could give the following geometric example instead. Consider an equilateral triangle ABC inscribed in a circle, as shown. P is a point on arc BC. Prove that $PA = PB + PC$.



The reader is probably already familiar with the famous case $x = 2$ known as the Pythagorean Theorem: A triangle with sides a, b, and c is a right-angled triangle if and only if the lengths satisfy $a^2 + b^2 = c^2$.



Of course, no discussion about algebraic relations in the above form is complete without mentioning the notorious Fermat's Last Theorem and the recent announcement that it has finally been laid to rest. If x is an integer with $x > 2$, then the claim is that no solution in the natural numbers exists for a, b, and c. However, in our general case, the values are real, so we are not limited by the above result to finding wild and wacky geometric or other interpretations for it.

If the reader is curious, as are we, then try the following exercises to find geometric interpretations for special cases of the above relation. The exercises are explorational and may not have nice solutions, if any. Readers are welcome to submit any interesting results they find.

Exercises

1. Let a, b, and c be the lengths of three line segments. Determine how these three line segments are related geometrically if they satisfy the relation

   (a) $c^3 = a^3 + b^3$,

   (b) $\sqrt{c} = \sqrt{a} + \sqrt{b}$,

   (c) $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

2. It is clear that algebraically, all the relations are similar. However, the geometric interpretations do not appear to be related. Is there a general geometric description where each of the above geometric figures is a special case?

3. The algebraic relation clearly does not work for the case $x = 0$. Is there a way to define the relation so that it would be consistent with everything else mentioned so far?

We extend congratulations to Ravi Vakil and Alice Staveley, who were married at St. John's, NF on Monday, 12 October 1998.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. We request that solutions from this issue be submitted by 1 September 1999, for publication in issue 8 of 1999.

High School Problems

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H245. Determine how many distinct integers there are in the set

$$\left\{\left\lfloor\frac{1^2}{1998}\right\rfloor, \left\lfloor\frac{2^2}{1998}\right\rfloor, \left\lfloor\frac{3^2}{1998}\right\rfloor, \ldots, \left\lfloor\frac{1998^2}{1998}\right\rfloor\right\}.$$

H246. Let $S(n)$ denote the sum of the first n positive integers. We say that an integer n is *fantastic* if both n and $S(n)$ are perfect squares. For example, 49 is fantastic, because $49 = 7^2$ and $S(49) = 1+2+3+\cdots+49 = 1225 = 35^2$ are both perfect squares. Find another integer $n > 49$ that is fantastic.

H247. Say that the integers a, b, c, d, p, and r form a cyclic set (a, b, c, d, p, r) if there exists a cyclic quadrilateral with circumradius r, sides a, b, c, and d, and diagonals p and $2r$.

(a) Show that if $r < 25$, no cyclic set exists.

(b) Find a cyclic set (a, b, c, d, p, r) for $r = 25$.

H248. Consider a tetrahedral die that has the four integers 1, 2, 3, and 4 written on its faces. Roll the die 2000 times. For each i, $1 \le i \le 4$, let $f(i)$ represent the number of times that i turned up. (So, $f(1) + f(2) + f(3) + f(4) = 2000$.) Also, let S denote the total sum of the 2000 rolls.

If $S^4 = 6144 \cdot f(1)f(2)f(3)f(4)$, determine the values of $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Advanced Problems

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A221. Construct, using straightedge and compass only, the common tangents of two non-intersecting circles.

A222. Does there exist a set of n consecutive positive integers such that for every positive integer $k < n$, it is possible to pick k of these numbers whose mean is still in the set?

A223. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose p is a prime with $p \equiv 3 \pmod 4$. Show that for any set of $p-1$ consecutive integers, the set cannot be divided into two subsets so that the product of the members of the one set is equal to the product of the members of the other set.

(Generalization of Question 4, IMO 1970)

A224. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let P be an interior point of triangle ABC such that $\angle PBA = \angle PCA = (\angle ABC + \angle ACB)/3$. Prove that

$$\frac{AC}{AB+PC} = \frac{AB}{AC+PB}.$$

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C81. Let $\{a_n\}$ be the sequence defined as follows: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$.

(a) Prove that $a_n^2 - a_{n-1}a_{n+1} = 1$ for all $n \geq 1$.

(b) Evaluate $\displaystyle\sum_{k=1}^{\infty} \arctan\left(\frac{1}{4a_k^2}\right)$.

C82. Find the smallest multiple of 1998 which appears as a partial sum of the increasing sequence

$$1,\ 1,\ 2,\ 2,\ 2,\ 4,\ 4,\ 4,\ 4,\ 8,\ \ldots,$$

in which the number 2^k appears $k+2$ times (for k a non-negative integer).

IMO Report

Adrian Chan, student, UCC, Toronto

This year's Canadian IMO team began with a week of training at the University of Calgary with lavish meal tickets. Then they were off to beautiful and rocky Kananaskis where we all had an "adventurous" time. Once the team stepped off the air-conditioned plane and into hot and muggy Taipei, Taiwan, it marked the team's official arrival to the 39^{th} International Mathematical Olympiad.

The team consisted of the following members: Adrian "Oops I dropped my..." Birka, Adrian "If You Will" Chan, Jimmy "Nuclear Aerial Strike" Chui, Mihaela "Baia" Enachescu, Jessie "So Cute" Lei, and Adrian "Nailing Radar" Tang. Team leader Dr. Christopher "Focus" Small was driven to the edge, while deputy leader J.P. "It's So Easy" Grossman calmly polished off old competitions one by one. Special thanks to leader observer Arthur "Rubik's Cube" Baragar and deputy observer Dorette "Dutch" Pronk for their coaching and experience. Also, thanks must go to Dr. Bill Sands of the University of Calgary for organizing such a fun training session.

The contest itself seemed to continue the trend of difficult IMO's and low medal cutoffs. With 76 countries competing, Canada fared extremely well, bringing back 1 gold, 1 silver, 2 bronze and an honourable mention. The scores were as follows:

CAN 1	Adrian Birka	10	
CAN 2	Adrian Chan	31	Gold Medal
CAN 3	Jimmy Chui	14	Bronze Medal
CAN 4	Mihaela Enachescu	30	Silver Medal
CAN 5	Jessie Lei	13	Honourable Mention
CAN 6	Adrian Tang	15	Bronze Medal

In this year's contest, Canada placed 20^{th} out of 76 countries, up from last year's 29^{th} ranking. Best of luck to CAN 1, 4, and 6 as they continue university studies at MIT, Harvard, and Waterloo respectively. CAN 2, 3, and 5 are all eligible for next year's team. Hopefully there won't be as much moaning of "Where did I go wrong?" next year around!

Special thanks must also go to Dr. Graham Wright of the Canadian Mathematical Society for again taking care of the tab, and Professor Ed Barbeau for his hard work and dedication to the training of potential IMO candidates through his year-long correspondence program.

Although sometimes things didn't make sense, and the IMO flag somehow disappeared, the 39^{th} International Mathematical Olympiad ran smoothly and was definitely a success. The new experience of a place halfway around the world with a stimulating culture was new to most of us. Best of luck to all IMO hopefuls for the 1999 team, as yet another Canadian IMO journey begins next July in Bucharest, Romania.

Bogus Arguments and Arcane Identities

Ravi Vakil
Princeton University

In Euler's time, mathematics was faster and looser than today, and niceties such as limits were blatantly ignored. Here is an argument of Euler's that seems to have no right to work, but does nonetheless. We conclude with some avenues for exploration and an open question.

If $r_1, \ldots, r_n$ are the zeros of a polynomial $a_0 + a_1 x + \cdots + a_n x^n$, and none of the r_i are zero (or $a_0 \neq 0$), then the negative of the linear term over the constant term is the sum of the reciprocals of the roots:

$$-\frac{a_1}{a_0} = \frac{1}{r_1} + \cdots + \frac{1}{r_n}.$$

What about power series? For example, $\cos x = 1 - x^2/2 + x^4/24 - \cdots$, so that

$$\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} - \cdots.$$

The zeroes of this function are the squares of the odd multiples of $\pi/2$: $((2n+1)\pi/2)^2$, $n = 0, 1, 2, \ldots$. One might hope that the principle for polynomials given above still holds:

$$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{1}{((2n+1)\pi/2)^2}$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \qquad (1)$$

This is actually true!

Another possibility is to use $\sin x$, which has power series expansion

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots.$$

Can you use the power series for $(\sin\sqrt{x})/x$ to "prove" the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\ ? \qquad (2)$$

Can you relate (2) to (1) by arguing that (2) minus a quarter of (2) is (1)?

If you write a short computer program to compute $\sum_{n=1}^{1000}\frac{1}{n^2}$, and compare it to $\pi^2/6$, you'll see that they are indeed very close. In fact, they differ by almost exactly $1/1000$. Can you explain why this might be? Can you guesstimate the difference between $\sum_{n=1}^{1000}\frac{1}{(2n+1)^2}$ and $\pi^2/8$?

And finally, can you conjure up other examples of this sort of argument, to "prove" other arcane identities? If so, please let us know!

Acknowledgements. This note was inspired by the example of $\cos\sqrt{x}$ given in [A].

References

[A] S. Abhyankar, *Historical ramblings in algebraic geometry and related algebra*, Amer. Math. Monthly **83** (1976), no. 6, 409–448.

[E] L. Euler, Introductio in Analysin Infinitorum, Berlin Academy, 1748.

The Fibonacci Sequence

Wai Ling Yee
student, University of Waterloo

The sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ is called the *Fibonacci sequence*. Named after Leonardo of Pisa, who is also known as Fibonacci (unsurprisingly), it is one of the most widely studied sequences of all time. The Fibonacci sequence is an excellent topic with which to begin learning some basic number theory and various techniques for working with recurrence relations.

Basic Results

Theorem 1. For all $n \geq 1$,

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}.$$

Proof by Induction. When $n = 1$, $F_1^2 - F_0F_2 = 1^2 - 0 \cdot 1 = 1 = (-1)^{1-1}$, so the formula holds for $n = 1$. Assume that the formula holds for some $n = k$, $k \geq 1$. For $n = k + 1$,

$$\begin{aligned}
F_{k+1}^2 - F_kF_{k+2} &= F_{k+1}^2 - F_k(F_{k+1} + F_k) \\
&= (F_{k+1} - F_k)F_{k+1} - F_k^2 \\
&= F_{k-1}F_{k+1} - F_k^2 \\
&= -(-1)^{k-1} \quad \text{by the induction hypothesis} \\
&= (-1)^k,
\end{aligned}$$

so the formula holds for $n = k + 1$. Therefore, by mathematical induction, the formula holds for all $n \geq 1$.

Theorem 2. For all $n \geq 1$,

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. Using the recurrence relation n times, we have

$$\begin{aligned}
F_{2n} &= F_{2n-1} + F_{2n-2} \\
&= F_{2n-1} + F_{2n-3} + F_{2n-4} \\
&= \cdots \\
&= F_{2n-1} + F_{2n-3} + \cdots + F_3 + F_1 + F_0 \\
&= F_{2n-1} + F_{2n-3} + \cdots + F_3 + F_1.
\end{aligned}$$

Exercise 1. Prove

$$\sum_{j=1}^{n} F_j^2 = F_n F_{n+1} .$$

Divisibility

Theorem 3. For all m, $n \geq 1$,

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m .$$

Proof. We will prove this by induction on m. When $m = 1$,

$$F_{n+1} = F_n \cdot 1 + F_{n-1} \cdot 1 = F_n F_2 + F_{n-1} F_1,$$

so the formula holds for all n when $m = 1$. Assume that the formula holds for all n when $m = M$. For $m = M + 1$,

$$\begin{aligned} F_{n+M+1} &= F_{n+M} + F_{(n-1)+M} \\ &= F_n F_{M+1} + F_{n-1} F_M + F_{n-1} F_{M+1} + F_{n-2} F_M \\ &\qquad \text{by the induction hypothesis} \\ &= F_n F_{M+1} + (F_{n-1} + F_{n-2}) F_M + F_{n-1} F_{M+1} \\ &= F_n (F_M + F_{M+1}) + F_{n-1} F_{M+1} \\ &= F_n F_{M+2} + F_{n-1} F_{M+1}, \end{aligned}$$

so the formula holds for all n for $m = M + 1$. By mathematical induction, the formula holds for all m, $n \geq 1$.

Corollary 4. For all m, $n \geq 1$, $F_n | F_{nm}$.

Proof. We will prove this by induction on m. When $m = 1$, F_n certainly divides itself for every positive integer n. Suppose the statement holds for all n when $m = M$. For $m = M + 1$,

$$F_{n(M+1)} = F_{nM+n} = F_{nM} F_{n+1} + F_{nM-1} F_n$$

by Theorem 3. Since F_{nM} is divisible by F_n by the induction hypothesis, $F_{nM} F_{n+1} + F_{nM-1} F_n$ is also divisible by F_n. This is equivalent to $F_n | F_{n(M+1)}$, so the result holds for $m = M + 1$. By mathematical induction, the formula holds for all m, $n \geq 1$.

Exercise 2. Prove that for every positive integer n, there exist n consecutive, composite Fibonacci numbers.

Number Theory 101

We will now define a few terms in the interests of formality. For integers a and b, we say that a *divides* b if there exists an integer q such that $b = aq$, and a is called a *divisor* of b. Given two non-zero integers a and b, the largest number which divides both of them, denoted $\gcd(a, b)$, is called their *greatest common divisor*. If $\gcd(a, b) = 1$, then a and b are said to be *relatively prime*.

Theorem 5. (The Division Algorithm) Given a positive integer a and an integer b, there exist unique integers q and r such that $b = aq + r$ and $0 \le r < a$. Then q is called the *quotient*, and r is called the *remainder* upon division of b by a.

Proof. Consider the set

$$S = \{s : s = b - aq \ge 0, q \in \mathbb{Z}\}.$$

S cannot be empty. If $b \ge 0$, then select $q = 0$ to give $b \in S$. Otherwise, if $b < 0$, select $q = b$ so that $b - ab = b(1 - a) \ge 0$, which means that $b - ab \in S$. Since S is non-empty and contains only non-negative integers, we can find the smallest element in S. Call it r.

Suppose $r \ge a$. Then $0 \le r - a = b - aq - a = b - a(q + 1)$ for some q, so $r - a \in S$ and it is smaller than r, contradiction. Thus $0 \le r < a$. Suppose that we can find $0 \le r_1 < r_2 < a$ in S and corresponding q_1 and q_2. Then $b = aq_1 + r_1 = aq_2 + r_2$, which implies that $a(q_1 - q_2) = r_2 - r_1$. Thus a divides $r_2 - r_1$. However, we also know that $0 < r_2 - r_1 < a$; that is, $r_2 - r_1$ lies between two consecutive multiples of a and thus cannot be divisible by a, contradiction. We have shown the existence and uniqueness of q and r.

Exercise 3. Prove that $\gcd(a, b) = \gcd(a, b - aq)$ for any non-zero integers a and b and any integer q.

Exercise 4. Prove that if a and q are relatively prime, then $\gcd(a, qb) = \gcd(a, b)$, where a, b, q are non-zero integers.

Exercise 5. Prove that F_n and F_{n+1} are relatively prime.

Number Theory 102

The Euclidean Algorithm. The Euclidean Algorithm is an algorithm used to determine the greatest common divisor of two numbers. Suppose we have two distinct positive integers a and b where, without loss of generality, $a < b$. By the Division Algorithm, $b = aq_1 + r_1$ where $0 \le r_1 < a$ for unique integers q_1 and r_1. If $r_1 = 0$, then a divides b so our greatest common divisor is a. Otherwise, by Exercise 3, $\gcd(a, b) = \gcd(a, b - aq_1) = \gcd(a, r_1) = \gcd(r_1, a)$. In this case, we then repeat the same argument using r_1 and a where we used a and b before, respectively. We have $a = r_1 q_2 + r_2$ where $0 \le r_2 < r_1$ for unique integers q_2 and r_2 by the Division Algorithm, and $\gcd(r_1, a) = \gcd(r_1, a - q_2 r_1) = \gcd(r_2, r_1)$. Continue applying this argument. Since the r_is are strictly decreasing and non-negative, there must be a last remainder, say r_n, that is bigger than 0. So we have

$$\begin{array}{rcllrcl} b &=& aq_1 + r_1, & 0 \le r_1 < a, & \gcd(a,b) &=& \gcd(r_1,a), \\ a &=& r_1 q_2 + r_2, & 0 \le r_2 < r_1, & \gcd(r_1,a) &=& \gcd(r_2,r_1), \\ r_1 &=& r_2 q_3 + r_3, & 0 \le r_3 < r_2, & \gcd(r_2,r_1) &=& \gcd(r_3,r_2), \\ \cdots & & & & & \cdots & \\ r_{n-2} &=& r_{n-1} q_n + r_n, & 0 \le r_n < r_{n-1}, & \gcd(r_{n-1},r_{n-2}) &=& \gcd(r_n,r_{n-1}), \\ r_{n-1} &=& r_n q_{n+1}, & & \gcd(r_n,r_{n-1}) &=& r_n. \end{array}$$

We have found that $\gcd(a,b) = r_n$.

Theorem 6. For all a, $b \ge 1$,

$$\gcd(F_a, F_b) = F_{\gcd(a,b)}.$$

Proof. If a and b are equal, the result is immediate, so assume that $a < b$. Apply the Euclidean Algorithm to obtain

$$\begin{array}{rcll} b &=& aq_1 + r_1, & 0 \le r_1 < a, \\ a &=& r_1 q_2 + r_2, & 0 \le r_2 < r_1, \\ r_1 &=& r_2 q_3 + r_3, & 0 \le r_3 < r_2, \\ & \cdots & & \\ r_{n-2} &=& r_{n-1} q_n + r_n, & 0 \le r_n < r_{n-1}, \\ r_{n-1} &=& r_n q_{n+1}. & \end{array}$$

We have

$$\gcd(F_a, F_b) \;=\; \gcd(F_a, F_{aq_1+r_1}) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1} + F_{aq_1}F_{r_1+1})$$

by Theorem 3. Since $F_{aq_1}F_{r_1+1}$ is a multiple of F_a by Corollary 4,

$$\gcd(F_a, F_b) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1} + F_{aq_1}F_{r_1+1}) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1})$$

by Exercise 3. By Exercise 5, $\gcd(F_{aq_1}, F_{aq_1-1}) = 1$. Since F_a divides F_{aq_1}, $\gcd(F_a, F_{aq_1-1}) = 1$ also. By Exercise 4, since F_a and F_{aq_1-1} are relatively prime,

$$\gcd(F_a, F_{aq_1-1}F_{r_1}) \;=\; \gcd(F_a, F_{r_1}).$$

We conclude that $\gcd(F_a, F_b) = \gcd(F_{r_1}, F_a)$. Repeating this argument, we obtain

$$\gcd(F_{r_1}, F_a) \;=\; \gcd(F_{r_2}, F_{r_1}) \;=\; \cdots \;=\; \gcd(F_{r_n}, F_{r_{n-1}}).$$

Since r_n divides r_{n-1}, F_{r_n} divides $F_{r_{n-1}}$, which implies that $\gcd(F_{r_n}, F_{r_{n-1}}) = F_{r_n}$. Thus,

$$\gcd(F_a, F_b) \;=\; F_{r_n} \;=\; F_{\gcd(a,b)}.$$

Finding F_n Explicitly

The monic quadratic with roots α and β is

$$(x-\alpha)(x-\beta) \;=\; x^2 - (\alpha+\beta)x + \alpha\beta.$$

Let α and β be the roots of $x^2 - x - 1$ in particular. Then, comparing coefficients, $\alpha + \beta = 1$ and $\alpha\beta = -1$. Using this, we can rewrite the recurrence relation $F_n = F_{n-1} + F_{n-2}$ as $F_n = (\alpha + \beta)F_{n-1} - \alpha\beta F_{n-2}$. From this equation, we obtain

$$F_n - \alpha F_{n-1} = \beta(F_{n-1} - \alpha F_{n-2}).$$

Let $s_{n-1} = F_n - \alpha F_{n-1}$ for all $n \geq 2$. Rewriting the above equation in terms of the s_i, we obtain $s_{n-1} = \beta s_{n-2}$. In other words, the sequence $\{s_n\}$ is a geometric sequence with common ration β. We conclude that $s_n = \beta^{n-1} s_1$.

Similarly, $F_n - \beta F_{n-1} = \alpha(F_{n-1} - \beta F_{n-2})$, and if we let $t_{n-1} = F_n - \beta F_{n-1}$ for $n \geq 2$, then $t_n = \alpha^{n-1} t_1$ for $n \geq 1$. Hence,

$$\begin{aligned}
F_n &= \frac{\alpha - \beta}{\alpha - \beta} F_n + \frac{\alpha\beta}{\alpha - \beta} F_{n-1} - \frac{\alpha\beta}{\alpha - \beta} F_{n-1} \\
&= \frac{\alpha(F_n - \beta F_{n-1}) - \beta(F_n - \alpha F_{n-1})}{\alpha - \beta} \\
&= \frac{\alpha t_{n-1} - \beta s_{n-1}}{\alpha - \beta} \\
&= \frac{\alpha^{n-1} t_1 - \beta^{n-1} s_1}{\alpha - \beta} \\
&= \frac{\alpha^{n-1}(F_2 - \beta F_1) - \beta^{n-1}(F_2 - \alpha F_1)}{\alpha - \beta} \\
&= \frac{\alpha^{n-1}(1 - \beta) - \beta^{n-1}(1 - \alpha)}{\alpha - \beta}.
\end{aligned}$$

Recall that $\alpha + \beta = 1$, so the equation above is, in fact,

$$\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Solving $x^2 - x - 1 = 0$ for the values of α and β, we have, without loss of generality,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Substituting these values, we conclude that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This is called *Binet's Formula* for the Fibonacci sequence.

Exercise 5. Let $\tau = (1 + \sqrt{5})/2$. Prove that F_n is the integer closest to $\dfrac{\tau^n}{\sqrt{5}}$.

Problems

1. Prove that the product of every four consecutive Fibonacci numbers is the area of a Pythagorean triangle.
2. Prove that every positive integer can be written as a sum of distinct Fibonacci numbers.
3. Prove
$$F_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1}.$$
4. Prove that if F_n is prime and $n \geq 5$, then n is prime.
5. Prove that $F_n + 1$ is always composite for $n \geq 4$.
6. Show that for any positive integer n, among the first n^2 Fibonacci numbers, there exists at least one that is divisible by n.
7. Define a Fibonacci prime to be a Fibonacci number that is prime. Prove or disprove: There are infinitely many Fibonacci primes. (Note that this is an open problem.)

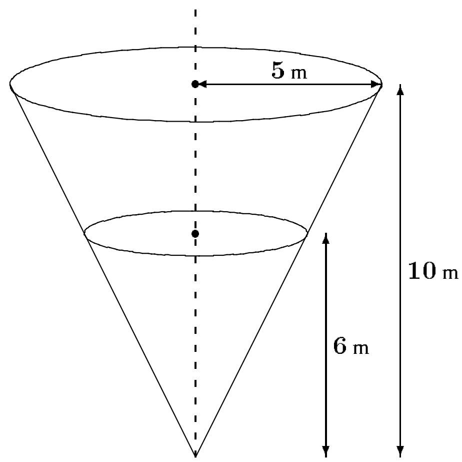
J.I.R. McKnight Problems Contest 1984

1. Find the real roots of the equation:

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}} \;=\; 1\,.$$

(Note: All square roots are to be taken as positive.)

2. Consider a reservoir in the shape of an inverted cone as shown in the diagram below. Water runs into the reservoir at the constant rate of $2\ \mathrm{m}^3$ per minute. How fast is the water level rising when it is 6 metres deep?



3. Three forces of magnitudes 10 N, 15 N, and 10 N act at angles of $30°$, $70°$, and $120°$ respectively, to the real axis O_x. Using the complex numbers and the imaginary axis O_y find the magnitude and direction of the resultant force.

4. Normals are drawn from the point $(\frac{15}{4}, -\frac{3}{4})$ to the parabola whose equation is $y^2 = 4x$. Find the coordinates of the points where the normals meet the parabola.

5. The horizontal base of a triangular pyramid is an equilateral triangle QRS, each of whose sides is 20 cm long. The sloping edges of the pyramid $PQRS$ are respectively 20 cm, 20 cm, and 12 cm long.

   (a) Calculate the perpendicular height of the pyramid to the nearest millimetre.

   (b) Calculate the angle of inclination of each of the three edges with the base to the nearest tenth of a degree.

6. Prove that if $\tan A = \tan^3 B$ and $\tan 2B = 2\tan C$, then $A+B-C = n\pi$ for some $n \in \mathbb{Z}$.

Swedish Mathematics Olympiad

1988 Qualifying Round

1. Show that the function

$$f(x) = \sqrt{x - 4\sqrt{x-1} + 3} + \sqrt{x - 6\sqrt{x-1} + 8}$$

is constant on the closed interval $5 \leq x \leq 10$.

2. Find the rational root of the equation

$$(2x)^{\log 2} = (3x)^{\log 3}, \quad x > 0$$

in the form $\frac{p}{q}$, where p and q are integers.

3. We will call two squares on a chessboard "neighbours" if they have a side or corner in common. The numbers 1 to 64 are arbitrarily placed on the 64 squares of a chessboard. Show that there are always two "neighbours" whose numbers have positive difference at least 9.

4. A car's tires wear proportionally with the distance driven. Furthermore, front tires last a km and back tires b km, where $a < b$. If, after an appropriate distance is driven, the tires are rotated (that is, back tires placed on front wheels and front tires on back wheels), the distance which can be driven without needing to replace any of the tires can be increased. What is the longest distance which can be driven with a set of tires, before any new tires must be bought?

5. P, Q, and R are points on the circumference of a circle such that PQR is an equilateral triangle. S is an arbitrary point on the circumference of the circle. Consider the lengths of the line segments PS, QS, and RS. Show that one of them is the sum of the other two.

6. Show that for every positive integer n, there exist positive integers x and y such that

$$\sqrt{x^2 + nxy + y^2}$$

is an integer.

1988 Final Round

1. The sides of a triangle have lengths $a > b > c$, and the corresponding perpendiculars have lengths h_a, h_b, and h_c. Show that

$$a + h_a \; > \; b + h_b \; > \; c + h_c \,.$$

2. Six ducklings swim on the surface of a pond, which is in the shape of a circle with radius 5 m. Show that, at every instant, two of the ducklings swim at a distance of at most 5 m from each other.

3. Show that for aribtrary real numbers x_1, x_2, and x_3,

   if $x_1 + x_2 + x_3 = 0$, then $x_1x_2 + x_2x_3 + x_3x_1 \;\leq\; 0\,.$

   Find all $n \geq 4$ for which the statement

   if $x_1 + x_2 + \cdots + x_n = 0$, then $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \leq 0$

   is true. (Both sums have n terms.)

4. Let $P(x)$ be a polynomial of degree 3 with exactly three distinct real roots. Find the number of real roots of the equation

$$(P'(x))^2 - 2P(x)P''(x) \;=\; 0\,.$$

5. Let m and n be positive integers. Show that there exists a constant $\alpha > 1$, independent of m and n, such that

$$\frac{m}{n} < \sqrt{7} \quad \text{implies that} \quad 7 - \frac{m^2}{n^2} \geq \frac{\alpha}{n^2}\,.$$

6. The sequence a_1, a_2, ... , is defined by the recursion formula

$$a_{n+1} \;=\; \sqrt{a_n^2 + \frac{1}{a_n}} \qquad n \geq 1\,,$$

   and $a_1 = 1$. Show that one can choose α such that

$$\frac{1}{2} \leq \frac{a_n}{n^\alpha} \leq 2 \qquad \text{for all } n \geq 1.$$