THE OLYMPIAD CORNER

No. 193

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first Olympiad to give you puzzling pleasure, we give the 18$^{\text{th}}$ Austrian-Polish Mathematics Competition written in Austria June 28–30, 1995. My thanks go to Bill Sands, University of Calgary, who collected this contest while assisting at the International Olympiad in Toronto in 1995, as well as to Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

18$^{\text{th}}$ AUSTRIAN-POLISH MATHEMATICS COMPETITION

Problems of the Individual Contest

June 28-29, 1995 (Time: 4.5 hours)

1. For a given integer $n \geq 3$ find all solutions $(a_1, \dots, a_n)$ of the system of equations

$$a_3 = a_2 + a_1, \quad a_4 = a_3 + a_2, \dots, a_n = a_{n-1} + a_{n-2}$$

$$a_1 = a_n + a_{n-1}, \quad a_2 = a_1 + a_n$$

in real numbers.

2. Let A_1, A_2, A_3, A_4 be four distinct points in the plane and let $X = \{A_1, A_2, A_3, A_4\}$. Show that there exists a subset Y of the set X with the following property: there is no disc K such that $K \cap X = Y$. *Note:* All points of the circle limiting a disc are considered to belong to the disc.

3. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

4. Determine all polynomials $P(x)$ with real coefficients, such that

$$(P(x))^2 + (P(1/x))^2 = P(x^2)P(1/x^2) \quad \text{for all} \quad x \neq 0.$$

5. An equilateral triangle ABC is given. Denote the mid-points of sides BC, CA, AB respectively by A_1, B_1, C_1. Three distinct parallel lines p, q, r are drawn through A_1, B_1, C_1, respectively. Line p cuts B_1C_1 at A_2;

line q cuts C_1A_1 at B_2; line r cuts A_1B_1 at C_2. Prove that the lines AA_2, BB_2, CC_2 concur at a point D lying on the circumcircle of triangle ABC.

6. The Alpine Club consisting of n members organizes four high-mountain expeditions for its members. Let E_1, E_2, E_3, E_4 be the four teams participating in these expeditions. How many ways are there to compose those teams, given the condition that $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, $E_3 \cap E_4 \neq \emptyset$?

Problems of the Team Contest

June 30, 1995 (Time: 4 hours)

7. For every integer c consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, with integer unknowns x and y. Determine all integers c for which the number of solutions (x, y) in pairs of integers satisfying the additional conditions (A) and (B) is a maximum:

(A) the number $|x|$ is the square of an integer;

(B) the number y is square-free (that is, there is no prime p with p^2 dividing y).

8. Consider the cube with vertices $\{\pm 1, \pm 1, \pm 1)$; that is, the set $\{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$. Let $V_1, \ldots, V_{95}$ be points of that cube. Denote by v_i the vector from $(0, 0, 0)$ to V_i. Consider the 2^{95} vectors of the form $s_1v_1 + s_2v_2 + \cdots + s_{95}v_{95}$, where $s_i = 1$ or $s_i = -1$.

(a) Let $d = 48$. Show that among all such vectors one can find a vector $w = (a, b, c)$ with $a^2 + b^2 + c^2 \leq d$.

(b) Find a number $d < 48$ with the same property.

Note: The smaller d, the better mark will be attracted by the solution.

9. Prove that the following inequality holds for all integers $n, m \geq 1$ and all positive real numbers x, y:

$$(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^n y^m + x^m y^n) \geq nm(x^{n+m-1}y + xy^{n+m-1}).$$

The next contest we give was also collected by Bill Sands while he was assisting at the IMO in Toronto. These are the problems of the 9th Iberoamerican Mathematical Olympiad held September 20, 21 in Fortaleza, Brazil. Students were given $4\frac{1}{2}$ hours each day.

9th IBEROAMERICAN MATHEMATICAL OLYMPIAD

Fortaleza, Brazil, September 20–21, 1994

First Day — Time: 4.5 hours

1. (Mexico): A natural number n is called *brazilian* if there exists an integer r, with $1 < r < n - 1$, such that the representation of the number

n in base r has all the digits equal. For example, 62 and 15 are brazilian, because 62 is written 222 in base 5 and 15 is 33 in base 4. Prove that 1993 is **not** brazilian, but 1994 is brazilian.

2. (Brazil): Let $ABCD$ be a cyclic quadrilateral. We suppose that there exists a circle with centre in AB, tangent to the other sides of the quadrilateral.

(i) Show that $AB = AD + BC$.

(ii) Calculate, in terms of $x = AB$ and $y = CD$, the maximal area that such a quadrilateral can reach.

3. (Brazil): In each cell of an $n \times n$ chessboard is a lamp. When a lamp is touched, the state of this lamp, and also the state of all the lamps in its row and in its column, is changed (switched from OFF to ON and vice versa). At the beginning, all the lamps are OFF. Show that it is always possible, with suitable sequence of touches, to turn ON all the lamps of the chessboard, and find, in terms of n, the minimal number of touches in order that all the lamps of the chessboard are ON.

Second Day — Time: 4.5 hours

4. (Brazil): The triangle ABC is acute, with circumcircle k. Let P be an internal point to k. The lines AP, BP, CP meet k again at X, Y, Z. Determine the point P for which triangle XYZ is equilateral.

5. (Brazil): Let n and r be two positive integers. We wish to construct r subsets of $\{0, 1, \ldots, n-1\}$, called $A_1, \ldots, A_r$, with $\text{card}(A_i) = k$ and such that, for each integer x, $0 \le x \le n-1$, there exist $x_1 \in A_1$, $x_2 \in A_2, \ldots, x_r \in A_r$ (an element in each subset), with

$$x = x_1 + x_2 + \cdots + x_r.$$

Find, in terms of n and r, the minimal value of k.

6. (Brazil): Show that all natural numbers $n \le 2^{1000000}$ can be obtained beginning at 1 with less than 1100000 sums; that is, there exists a finite sequence of natural numbers $x_0, x_1, \ldots, x_k$, with $k < 1100000$, $x_0 = 1$, $x_k = n$, such that for each $i = 1, 2, \ldots, k$, there exists r, s, with $0 \le r < i$, $0 \le s < i$, and $x_i = x_r + x_s$.

As a final problem set to challenge you, we present the problems of the IX, X and XI Grade of the Georgian Mathematical Olympiad, Final Round for 1995. It is interesting that 60% of the Grade XI problems come from the Grade IX paper. My thanks again go to Bill Sands, University of Calgary, for collecting these problems while he assisted with the IMO in Toronto in 1995.

GEORGIAN MATHEMATICAL OLYMPIAD 1995
Final Round
GRADE IX

1. A three-digit number was decreased by the sum of its digits. Then the result was decreased by the sum of *its* digits and so on. Show that on the 100^{th} step of this procedure the result will be zero, whatever the initial three-digit number is chosen. How many repetitions are enough to get zero?

2. Two circles of the same size are given. Seven arcs, each of them of 3° measure, are taken on the first circle and 10 arcs, each of them of 2° measure, are taken on the second one. Prove that it is possible to place one circle on the other so that these arcs do not intersect. Is it or is it not possible to prove the same if the number of arcs with measure 2° is 11?

3. Prove that if the product of three positive numbers is 1 and their sum is more than the sum of their reciprocals, then only one of these numbers can be more than 1.

4. Prove that in any convex hexagon there exists a diagonal which cuts from the hexagon a triangle with area less than $\frac{1}{6}$ of the area of the hexagon.

5. The set M of integers has the following property: if the numbers a and b are in M, then $a + 2b$ also belongs to M. It is known that the set contains positive as well as negative numbers. Prove that if the numbers a, b and c are in M, then $a + b - c$ is also in M.

GRADE X

1. (a) Five different numbers are written in one line. Is it always possible to choose three of them placed in increasing or decreasing order?

(b) Is it always possible to do the same, if we have to choose four numbers from nine?

2. (Same as IX.2)

3. Prove that for any natural number n, the average of all its factors lies between the numbers $\sqrt{n}$ and $\frac{n+1}{2}$.

4. The incircle of a triangle divides one of its medians into three equal parts. Find the ratio of the sides of the triangle.

5. The function f is given on the segment $[0, 1]$. It is known that $f(x) \geq 0$ and $f(1) = 1$. Besides that, for any two numbers x_1 and x_2, if $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 \leq 1$, then $f(x_1 + x_2) \geq f(x_1) + f(x_2)$.

(a) Prove that $f(x) \leq 2x$ for any x.

(b) Does the inequality $f(x) \leq 1.9x$ hold for every x?

GRADE XI

1. (Same as IX.3)

2. (Same as IX.1)

3. How many solutions has the equation $x = 1995 \sin x + 199$?

4. (Same as IX.4)

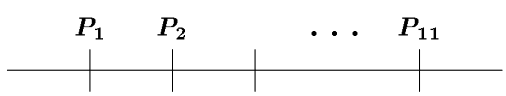
5. A natural number is written in each square of an $m \times n$ rectangular table. By one move, it is allowed to double all numbers of any row or subtract 1 from all numbers of any column. Prove that by repeating these moves several times, all numbers in the table become zeros.

Next a bit of housekeeping. After the columns were set, and before they appeared in print form, I received solutions from Pavlos Maragoudakis to the six problems of the Swedish contest for which we published the solutions last issue [1997: 196; 1998: 328–329]. He also submitted solutions to problems 1, 2, 3 and 5 of the Dutch Mathematical Olympiad, Second Round, 1993 [1997: 197; 1998: 329–332]. Last issue we gave solutions to all but the last problem. It was rather unfortunate timing in terms of acknowledging his contribution, but we are able to close out the file by having a complete set of solutions from the readers.

5. $P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line. $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances P_iP_j, $1 \leq i < j \leq 11$ is smaller than 30.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Without loss of generality we suppose that $P_1, P_2, \ldots, P_{11}$ are adjacent.



Now if $1 \leq i < j \leq 11$, then $P_iP_j = P_jP_1 - P_iP_1$. So

$$\begin{aligned}
\sum_{1 \leq i < j \leq 11} P_iP_j &= \sum_{1 \leq i < j \leq 11} (P_jP_1 - P_iP_1) \\
&= 10P_{11}P_1 + 9P_{10}P_1 - P_{10}P_1 + 8P_9P_1 - 2P_9P_1 \\
&\quad +7P_8P_1 - 3P_8P_1 + 6P_7P_1 - 4P_7P_1 + 5P_6P_1 \\
&\quad -5P_6P_1 + 4P_5P_1 - 6P_5P_1 + 3P_4P_1 - 7P_4P_1 \\
&\quad +2P_3P_1 - 8P_3P_1 + P_2P_1 - 9P_2P_1
\end{aligned}$$

$$\begin{aligned} &= 10P_{11}P_1 + 8P_{10}P_1 + 6P_9P_1 + 4P_8P_1 + 2P_7P_1 \\ &\quad -2P_5P_1 - 4P_4P_1 - 6P_3P_1 - 8P_2P_1 \\ &= 10P_{11}P_1 + 8(P_{10}P_1 - P_2P_1) + 6(P_9P_1 - P_3P_1) \\ &\quad +4(P_8P_1 - P_4P_1) + 2(P_7P_1 - P_5P_1) \\ &= 10P_{11}P_1 + 8P_{10}P_2 + 6P_9P_3 + 4P_8P_4 + 2P_7P_5 \\ &< 10\cdot 1 + 8\cdot 1 + 6\cdot 1 + 4\cdot 1 + 2\cdot 1 = 30 \end{aligned}$$

Also setting the record straight, I found amongst the solutions for another contest, the solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain to problem 2 of the Dutch Mathematical Olympiad, Second Round, for which we published a solution last issue [1997: 197, 1998: 330–331]. My apologies.

While we do not normally give solutions to problems of the USAMO, I am giving two comments/solutions from our readers to problems of the USAMO 1997 [1997: 261, 262].

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

Comment by Mansur Boase, student, St. Paul's School, London, England.

The result is immediate from Steiner's Theorem:

If the perpendiculars from the vertices A, B, C of a triangle ABC to the sides B_1C_1, C_1A_1, and A_1B_1, respectively, of a second triangle A_1, B_1, C_1 are concurrent, then the perpendiculars from the vertices A_1, B_1, C_1 of the triangle $A_1B_1C_1$ to the sides BC, CA, AB are also concurrent.

5. Prove that for positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \le (abc)^{-1}.$$

Solutions by Mansur Boase, student, St. Paul's School, London, England and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's presentation.

Since the inequality is homogeneous, we can assume $abc = 1$. Then if we let $x = a^3$, $y = b^3$, $z = c^3$, the inequality becomes

$$\frac{1}{1+x+y} + \frac{1}{1+y+z} + \frac{1}{1+z+x} \le 1 \qquad (1)$$

where $xyz = 1$ and x, y, z are positive. On expanding, (1) is equivalent to

$$(x+y+z)(xy+yz+zx-2) \ge 3\,.$$

This follows from the known elementary inequalities

$$\frac{x+y+z}{3} \geq \left(\frac{yz+zx+xy}{3}\right)^{1/2} \geq (xyz)^{1/3}.$$

There is equality if and only if $x = y = z = 1$.

Comment: The inequality in the form (1) was also given in the Spring 1997, Senior A-Level Tournament Of The Towns competition. A generalization to

$$\frac{1}{1+S-x_1} + \frac{1}{1+S-x_2} + \cdots + \frac{1}{1+S-x_n} \leq 1,$$

where $S = x_1 + x_2 + \cdots + x_n$, $x_1 x_2 \ldots x_n = 1$, and $x_i > 0$ is due to Dragos Hrimiuc, University of Alberta, and will probably appear as a problem in Math. Magazine.

Next we give two solutions by our readers to two problems of the 3rd Ukrainian Mathematical Olympiad, March 26–27, 1994 given in [1997: 262].

2. (9–10) A convex polygon and point O inside it are given. Prove that for any $n > 1$ there exist points $A_1, A_2, \ldots, A_n$ on the sides of the polygon such that $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = \overrightarrow{0}$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

It follows by continuity that there always exists a chord A_1OA_1' such that $A_1O = A_1'O$ and hence $\overrightarrow{OA_1} + \overrightarrow{OA_1'} = \overrightarrow{0}$. Similarly, there exists a chord A_2A_3' which is bisected by the midpoint O_1 of OA_1'. It follows by the parallelogram law that $\overrightarrow{OA_2} + \overrightarrow{OA_3'} = \overrightarrow{OA_1'}$ and hence $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3'} = \overrightarrow{0}$. Again similarly there exists a chord A_3A_4' which is bisected by the midpoint of OA_3' so that $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4'} = \overrightarrow{0}$, and so on for any number of vectors $n > 1$.

3. (10) A sequence of natural numbers a_k, $k \geq 1$, such that for each k, $a_k < a_{k+1} < a_k + 1993$ is given. Let all prime divisors of a_k be written for every k. Prove that we receive an infinite number of different prime numbers.

Solution by Pavlos Maragoudakis, Pireas, Greece.

We suppose that there is a sequence of natural numbers such that $a_k < a_{k+1} < a_k + 1993$, $k \geq 1$, and the set of all prime divisors of all a_k is finite. Let $p_1, p_2, \ldots, p_r$ list all the prime divisors of all a_k. Now every a_k has the form $p_1^{a_1} \ldots p_r^{a_r}$, $a_i = 0, 1, 2, \ldots$, $i = 1, \ldots, r$.

Let $S = \{p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \mid a_i = 0, 1, 2, \ldots, \ i = 1, 2, \ldots, r\}$.

Define (x_n) with $x_1 < x_2 < \ldots$ such that $S = \{x_n \ / \ n \in \mathbb{N}^*\}$. We have that $a_k < a_{k+1}$ and $a_k \in S$, $k \geq 1$. Thus (a_k) is a subsequence of

(x_k). But $a_k < a_{k-1} + 1993 < a_{k-2} + 2 \cdot 1993 < \cdots < a_1 + (k-1)1993 < k(a_1 + 1993), k \geq 1$.

Hence $a_k < k(a_1 + 1993)$, $k \geq 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{a_n} > \sum_{n=1}^{\infty} \frac{1}{n(a_1 + 1993)} = \frac{1}{a_1 + 1993} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

and
$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{1}{x_n} = \sum_{a_1, \ldots, a_r \geq 0} \frac{1}{p_1^{\alpha_1} p_2^{a_2} \cdots p_r^{a_4}}$$
$$= \left(\sum_{a_1=0}^{\infty} \frac{1}{p_1^{\alpha_1}} \right) \left(\sum_{a_2=0}^{\infty} \frac{1}{p_2^{a_2}} \right) \cdots \left(\sum_{\alpha_r=0}^{\infty} \frac{1}{p_r^{a_r}} \right)$$
$$= \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdots \frac{p_4}{p_4 - 1} < +\infty,$$

which is a contradiction.

We now turn our attention to the solutions by readers to problems of the Mock Test of the Hong Kong Committee for the IMO 1994 [1997: 322–323].

INTERNATIONAL MATHEMATICAL OLYMPIAD 1994

Hong Kong Committee — Mock Test, Part I

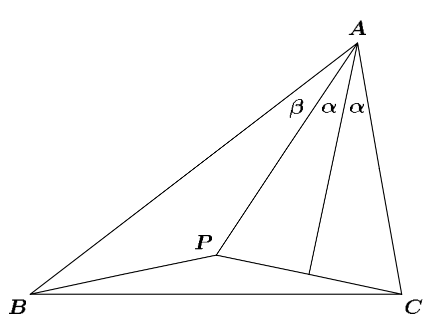
Time: 4.5 hours

1. In a triangle $\triangle ABC$, $\angle C = 2\angle B$. P is a point in the interior of $\triangle ABC$ satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution by Amengual Covas.

Let $\angle PAC$ and $\angle BAP$ be 2α and β respectively. Then, since $\angle C = 2\angle B$, we deduce from $A + B + C = 180°$ that

$$2\alpha + \beta + 3B = 180°. \qquad (1)$$



The angles at the base of the isosceles triangle PAC are each $90° - \alpha$. Also $\triangle BPC$ is isosceles, having base angles

$$C - (90° - \alpha) = 2B + \alpha - 90°,$$

and so

$$\begin{aligned} \angle BPA &= 180° - (\angle PBA + \angle BAP) \\ &= 180° - [B - (2B + \alpha - 90°) + 180° - 2\alpha - 3B] \\ &= 4B + 3\alpha - 90°. \end{aligned}$$

As usual, let a, b and c denote the lengths of the sides BC, AC and AB. By the Law of Cosines, applied to $\triangle BPA$, where $\overline{PA} = b$ and $\overline{PB} = \overline{PC} = 2b\sin\alpha$,

$$c^2 = b^2 + (2b\sin\alpha)^2 - 2\cdot b\cdot 2b\sin\alpha\cdot\cos(4B + 3\alpha - 90°)\,,$$

so that

$$c^2 = b^2[1 + 4\sin^2\alpha - 4\sin\alpha\sin(4B + 3\alpha)]\,. \tag{2}$$

We now use the fact that $\angle C = 2\angle B$ is equivalent to the condition $c^2 = b(b + a)$, which has appeared before in ***CRUX*** [1976: 74], [1984: 278] and [1996: 265–267]. Since $a = 2\cdot\overline{PC}\cdot\cos(2B + \alpha - 90°) = 4b\sin\alpha\sin(2B + \alpha)$, we have

$$c^2 = b^2[1 + 4\sin\alpha\sin(2B + \alpha)]\,. \tag{3}$$

Therefore, from (2) and (3), we get

$$b^2[1 + 4\sin^2\alpha - 4\sin\alpha\sin(4B + 3\alpha)] = b^2[1 + 4\sin\alpha\sin(2B + \alpha)]\,,$$

which simplifies to

$$\sin\alpha - \sin(4B + 3\alpha) = \sin(2B + \alpha)\,.$$

Since $\sin\alpha - \sin(4B + 3\alpha) = -2\cos(2B + 2\alpha)\sin(2B + \alpha)$, this equation may be rewritten as

$$\sin(2B + \alpha)\cdot[1 + 2\cos(2B + 2\alpha)] = 0.$$

Since, from (1), $2B + \alpha < 180°$, we must have $1 + 2\cos(2B + 2\alpha) = 0$, giving $\cos(2B + 2\alpha) = -1/2$; that is,

$$2B + 2\alpha = 120° \tag{4}$$

since, again from (1), $2B + 2\alpha < 180°$.

Finally, we may eliminate B between (1) and (4) to obtain $\alpha = \beta$. The result follows.

Mock Test, Part II

Time: 4.5 hours

1. Suppose that $yz + zx + xy = 1$ and x, y, and $z \geq 0$. Prove that

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \;\leq\; \frac{4\sqrt{3}}{9}.$$

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Klamkin.

We first convert the inequality to the following equivalent homogeneous one:

$$x(T_2-y^2)(T_2-z^2) + y(T_2-z^2)(T_2-x^2) + z(T_2-x^2)(T_2-y^2) \\ \leq\; (4\sqrt{3}/9)(T_2)^{5/2}$$

where $T_2 = yz+zx+xy$, and for subsequent use $T_1 = x+y+z$, $T_3 = xyz$. Expanding out, we get

$$T_1T_2^2 - T_2\sum x(y^2+z^2) + T_2T_3 \;\leq\; (4\sqrt{3}/9)(T_2)^{5/2},$$

or

$$T_1T_2^2 - T_2(T_1T_2 - 3T_3) + T_2T_3 = 4T_2T_3 \;\leq\; (4\sqrt{3}/9)(T_2)^{5/2}.$$

Squaring, we get one of the known Maclaurin inequalities for symmetric functions:

$$\sqrt[3]{T_3} \;\leq\; \sqrt[2]{T_2/3}\,.$$

There is equality if and only if $x = y = z$.

To finish this number of the *Corner* we give two solutions to problems of the 45$^{\text{th}}$ Mathematical Olympiad in Poland, Final Round [1997: 323–324].

1. Determine all triples of positive rational numbers (x, y, z) such that $x+y+z$, $x^{-1}+y^{-1}+z^{-1}$ and xyz are integers.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let $x+y+z = n_1$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = n_2$, and $xyz = n_3$, where n_1, n_2, n_3 are integers. Then $yz+zx+xy = n_2n_3$ and x, y, z are roots of the cubic

$$t^3 - n_1t^2 + n_2n_3t - n_3 \;=\; 0\,.$$

As known, the only rational roots of the latter are factors of n_3, and consequently x, y, z are integers.

The only triples of integers (x, y, z), aside from permutations, which satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = n_2$ are

$$(1,1,1),\ (1,2,2),\ (2,3,6),\ (2,4,4), \quad \text{and} \quad (3,3,3).$$

5. Let $A_1, A_2, \dots, A_8$ be the vertices of a parallelepiped and let O be its centre. Show that

$$4(OA_1^2 + OA_2^2 + \cdots + OA_8^2) \ \le\ (OA_1 + OA_2 + \cdots + OA_8)^2 .$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let one of the vertices be the origin and let the vectors $B + C$, $C + A$, $A + B$ denote the three coterminal edges emanating from this origin. Then the vectors to the remaining four vertices are $S + A$, $S + B$, $S + C$, and $2S$ where $S = A + B + C$ and which is also the vector to the centre. The inequality now becomes

$$2(S^2 + A^2 + B^2 + C^2) \ \le\ (|S| + |A| + |B| + |C|)^2 ,$$

or

$$S^2 + A^2 + B^2 + C^2 \ \le\ 2|S|\{|A| + |B| + |C|\} + 2\{|B|\,|C| + |C|\,|A| + |A|\,|B|\}.$$

Since

$$S^2 \ =\ A^2 + B^2 + C^2 + 2B \cdot C + 2C \cdot A + 2A \cdot B,$$

the inequality now becomes

$$S^2 - B \cdot C - C \cdot A - A \cdot B \ \le\ |S|\{|A| + |B| + |C|\} + \{|B|\,|C| + |C|\,|A| + |A|\,|B|\}.$$

Clearly,

$$S^2 \ \le\ |S|\{|A| + |B| + |C|\}$$

and

$$-B \cdot C - C \cdot A - A \cdot B \ \le\ |B|\,|C|\,|A| + |A|\,|B| .$$

There is equality if and only if the parellelepiped is degenerate, for example, $B = C = O$.

That completes this number of the *Olympiad Corner*. Send me your nice solutions and Olympiad contests.