THE OLYMPIAD CORNER

No. 179

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Another year has passed, and the first with Bruce Shawyer as Editor-in-Chief. He has made the transition pleasant and easy. Special thanks go to Joanne Longworth whose TEX skills have made changed fonts and formats easy to incorporate. Thanks also go to the many contributors to the two Corners including:

Miguel Amengual Covas	Cyrus C. Hsia	Dieter Ruoff
Sěfket Arslanagić	Murray Klamkin	Toshio Seimiya
Mansur Boase	Derek Kisman	Michael Selby
Seung-Jin Bang	Ted Lewis	D.J. Smeenk
Christopher Bradley	Joseph Ling	Daryl Tingley
Francisco Bellot Rosado	Beatriz Margolis	Panos E. Tsaoussoglou
Paul Colucci	Stewart Metchette	Ravi Vakil
Hans Engelhaupt	Richard Nowakowski	Edward T.H. Wang
Tony Gardiner	Michael Nutt	Hoe Teck Wee
Solomon Golomb	Siu Taur Pang	Chris Wildhagen
Gareth Griffith	Bob Prielipp	Siming Zhan
Georg Gunther	Chandan Reddy	

Let me open with a major apology. In the November 1996 number of the corner we gave twelve of the problems proposed to the jury but not used at the 36th International Mathematical Olympiad held in Canada. Those familiar with the process of selection will know that the problems do not initiate with the host country. They come from proposers in other countries, and the responsibility of the host country selection committee is to refine and select from these submissions the official list of problems proposed to the jury. Over the years I have loosely referred to the problems proposed to the jury at the Xth International Olympiad in Y as the "Y-problems" for short. However, when that became part of a longer more official sounding sub-title "Canadian Problems for consideration by the International Jury," I should have seen that it read as if the original proposers are from Canada, thus insulting the creators. In retrospect I do not understand why that interpretation did not jump off the page. To the many creative non-Canadians who submitted problems for possible use at the 36th IMO my sincere apologies.

As an Olympiad Contest this issue we give the problems of the 44th Mathematical Olympiad from Latvia. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong for collecting the problems for me.

LATVIAN 44 MATHEMATICAL OLYMPIAD

Final Grade, 3rd Round

Riga, 1994

1. It is given that $\cos x = \cos y$ and $\sin x = -\sin y$. Prove that $\sin 1994x + \sin 1994y = 0$.

2. The plane is divided into unit squares in the standard way. Consider a pentagon with all its vertices at grid points.

(a) Prove that its area is not less than $3/2$.

(b) Prove that its area is not less than $5/2$, if it is given that the pentagon is convex.

3. It is given that $a > 0$, $b > 0$, $c > 0$, $a + b + c = abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

4. Solve the equation $1! + 2! + 3! + \cdots + n! = m^3$ in natural numbers.

5. There are 1994 employees in the office. Each of them knows 1600 of the others. Prove that we can find 6 employees, each of them knowing all 5 others.

1st SELECTION ROUND

1. It is given that x and y are positive integers and $3x^2 + x = 4y^2 + y$. Prove that $x - y$, $3x + 3y + 1$ and $4x + 4y + 1$ are squares of integers.

2. Is it possible to find 2^{1994} different pairs of natural numbers (a_i, b_i) such that the following 2 properties hold simultaneously:

(1) $\dfrac{1}{a_1 b_1} + \dfrac{1}{a_2 b_2} + \cdots + \dfrac{1}{a_{2^{1994}} b_{2^{1994}}} = 1$,

(2) $(a_1 + a_2 + \cdots + a_{2^{1994}}) + (b_1 + b_2 + \cdots + b_{2^{1994}}) = 3^{1995}$?

3. A circle with unit radius is given. A system of line segments is called a cover iff each line with a common point with the circle also has some common point with some of the segments of the system.

(a) Prove that the sum of the lengths of the segments of a cover is more than 3,

(b) Does there exist a cover with this sum less than 5?

4. A natural number is written on the blackboard. Two players move alternatively. The first player's move consists of replacing the number n on the blackboard by $n/2$, by $n/4$ or by $3n$ (first two choices are allowed only if they are natural numbers). The second player's move consists of replacing the number n on the blackboard by $n + 1$ or by $n - 1$. The first player wants the number 3 to appear on the blackboard (no matter who writes it down). Can he always achieve his aim?

5. Three equal circles intersect at the point O and also two by two at the points A, B, C. Let T be the triangle whose sides are common tangents of the circles; T contains all the circles inside itself. Prove that the area of T is not less than 9 times the area of ABC.

2nd SELECTION ROUND

1. It is given that $0 \le x_i \le 1$, $i = 1, 2, \dots, n$. Find the maximum of the expression

$$\frac{x_1}{x_2x_3\dots x_n + 1} + \frac{x_2}{x_1x_3x_4\dots x_n + 1} + \dots + \frac{x_n}{x_1x_2\dots x_{n-1} + 1}.$$

2. There are $2n$ points on the circle dividing it into $2n$ equal arcs. We must draw n chords having these points as endpoints so that the lengths of all chords are different. Is it possible if:

(a) $n = 24$,

(b) $n = 1994$?

3. A triangle ABC is given. From the vertex B, n rays are constructed intersecting the side AC. For each of the $n+1$ triangles obtained, an incircle with radius r_i and excircle (which touches the side AC) with radius R_i is constructed. Prove that the expression

$$\frac{r_1r_2\dots r_{n+1}}{R_1R_2\dots R_{n+1}}$$

depends on neither n nor on which rays are constructed.

3rd SELECTION ROUND

1. A square is divided into n^2 cells. Into some cells "1" or "2" is written so that there is exactly one "1" and exactly one "2" in each row and in each column. We are allowed to interchange two rows or two columns; this is called a move. Prove that there is a sequence of moves such that after performing it "1"-s and "2"-s have interchanged their positions.

2. Let a_{ij} be integers, $|a_{ij}| < 100$. We know that the equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz = 0$$

has a solution $(1234, 3456, 5678)$. Prove that this equation also has a solution with x, y, z pairwise relatively prime which is not proportional to the given one.

3. Let $ABCD$ be an inscribed quadrilateral. Its diagonals intersect at O. Let the midpoints of AB and CD be U and V. Prove that the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent.

Next we give five Klamkin Quickies. My thanks go to Murray Klamkin, the University of Alberta, for supplying them to us. Next issue we will give the "quick" solutions along with another five of his special teasers.

FIVE KLAMKIN QUICKIES
October 21, 1996

1. For $x, y, z > 0$, prove that

(i) $1 + \dfrac{1}{(x+1)} \geq \left\{1 + \dfrac{1}{x(x+2)}\right\}^x$,

(ii) $[(x+y)(x+z)]^x[(y+z)(y+x)]^y[(z+x)(z+y)]^z \geq [4xy]^x[4yz]^y[4zx]^z$.

2. If $ABCD$ is a quadrilateral inscribed in a circle, prove that the four lines joining each vertex to the nine point centre of the triangle formed by the other three vertices are concurrent.

3. How many six digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square?

4. Let V_iW_i, $i = 1, 2, 3, 4$, denote four cevians of a tetrahedron $V_1V_2V_3V_4$ which are concurrent at an interior point P of the tetrahedron. Prove that

$$PW_1 + PW_2 + PW_3 + PW_4 \leq \max V_iW_i \leq \textbf{longest edge.}$$

5. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d, respectively.

We now turn to solutions from the readers to problems posed in the May 1995 number of the corner on the Sixth Irish Mathematical Olympiad, May 8, 1993 [1995: 151–152].

SIXTH IRISH MATHEMATICAL OLYMPIAD
May 8, 1993 — First Paper
(Time: 3 hours)

1. The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0, \qquad \beta^3 - 3\beta^2 + 5\beta + 11 = 0.$$

Find $\alpha + \beta$.

Solutions by Šefket Arslanagić, Berlin, Germany; by Beatriz Margolis, Paris, France; by Vedula N. Murty, Andhra University, Visakhapatnam, India; by D.J. Smeenk, Zaltbommel, the Netherlands; by Panos E. Tsaoussoglou, Athens, Greece; and comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Margolis's solution.

Define $f(x) = x^3 - 3x^2 + 5x$. We show that if $f(\alpha) + f(\beta) = 6$, then $\alpha + \beta = 2$. Since $f(x) = (x-1)^3 + 2(x-1) + 3$, we have

$$\begin{aligned} f(\alpha) - 3 &= (\alpha - 1)^3 + 2(\alpha - 1) \\ f(\beta) - 3 &= (\beta - 1)^3 + 2(\beta - 1) \end{aligned}$$

Adding gives

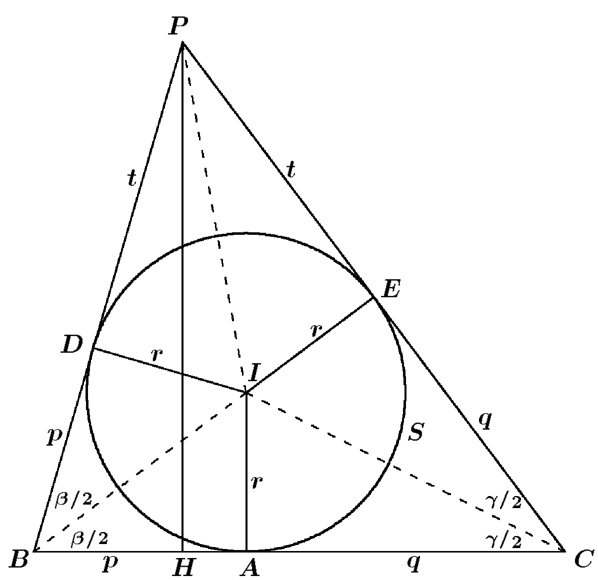
$$\begin{aligned} 0 &= (\alpha - 1)^3 + (\beta - 1)^3 + 2(\alpha + \beta - 2) \\ &= (\alpha + \beta - 2)[(\alpha - 1)^2 + (\alpha - 1)(\beta - 1) + (\beta - 1)^2 + 2] \end{aligned}$$

and, since the second factor is positive, we obtain the result. (See Olympiad Corner 142 and its solution.)

[*Wang's Comment:*] The problem is strikingly similar to problem 11.2 of the XXV Soviet Mathematical Olympiad, 11th Form [1993: 37]. The method used by Bradley given in the published solution [1994: 99] works for the present problem as well and, in fact, yields the same answer: $\alpha + \beta = 2$.

3. The line l is tangent to the circle S at the point A; B and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB| \cdot |AC|$ is constant, find the locus of P.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by D.J. Smeenk, Zaltbommel, the Netherlands. We use Smeenk's solution.



Let S be the incircle $s(I, 2)$ of $\triangle BCP$. We denote $\angle PBA = \beta$, $\angle PCA = \gamma$

$$\overline{AB} = p\overline{AC} = q$$

with $pq = k^2$, a constant.

Let S touch BP and CP at D and E respectively. For $\triangle PEI$ we have $\angle EIP = \frac{1}{2}(\beta+\gamma)$. Thus

$$t = r\tan\frac{1}{2}(\beta+\gamma) = \frac{(p+q)r^2}{pq-r^2}.$$

The semiperimeter of $\triangle BCP$ is

$$p+q+t = p+q+\frac{(p+q)r^2}{pq-r^2} = \frac{pq(p+q)}{pq-r^2}.$$

The area, F, of $\triangle BCP$ is

$$r\frac{pq(p+q)}{pq-r^2} = \frac{1}{2}(p+q)PH,$$

where PH is the altitude to BC. It follows immediately that

$$PH = \frac{2pqr}{pq-r^2} = \frac{2k^2r}{k^2-r}.$$

So the locus of P is a line parallel to BC.

4. Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $1/2^n$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ where the α_i denote the n real roots of f, $i = 1, 2, \ldots, n$. Then from $|f(0)| = f(1)$ we get $\prod(1-\alpha_i) = \prod \alpha_i$. (All products are over $i = 1, 2, \ldots, n$.) Using the Arithmetic Mean–Geometric Mean Inequality we then get

$$\left(\prod \alpha_i\right)^2 = \prod \alpha_i(1-\alpha_i) \;\leq\; \prod\left(\frac{\alpha_i + (1-\alpha_i)}{2}\right)^2 \;=\; \frac{1}{2^{2n}}$$

from which $\prod \alpha_i \leq \frac{1}{2^n}$ follows. Equality holds if and only if $\alpha_i = \frac{1}{2}$ for all $i = 1, 2, \ldots, n$.

May 8, 1993 — Second Paper

(Time: 3 hours)

3. For non-negative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty}\binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$$

for all integers n, r with $1 \leq r \leq n$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We use a combinatorial argument to establish the obviously equivalent identity

$$\sum_{d=1}^{k} \binom{n-r+1}{d}\binom{r-1}{r-d} = \binom{n}{r} \qquad (*)$$

where $k = \min\{r, n-r+1\}$. It clearly suffices to demonstrate that the left hand side of $(*)$ counts the number of ways of selecting r objects from n distinct objects (without replacements). Let $|S_2| = r-1$. For each fixed $d = 1, 2, \ldots, k$, any selection of d objects from S_1 $(S \backslash S_2)$ together with any selection of $r-d$ objections from S_2 would yield a selection of r objects from S. The total number of such selections is $\binom{n-r+1}{d}\binom{r-1}{r-d}$. Conversely, each selection of r objects from S clearly must arise in this manner. Summing over $d = 1, 2, \ldots,$ $(*)$ follows.

4. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Solutions by Šefket Arslanagić, Berlin, Germany; and by Vedula N. Murty, Andhra University, Visakhapatnam, India. We give Murty's solution.

We use mathematical induction. Let

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{(2k-1)}.$$

$S_1(x) = \sin x > 0$ for $x \in (0, \pi)$. Thus the proposed inequality is true for $n = 1$. Let $S_r(x) > 0$ for $r = 1, 2, \ldots, n-1$. We will deduce that $S_n(x) > 0$ for $x \in (0, \pi)$. Suppose that $S_n(x_0) \leq 0$ for some $x_0 \in (0, \pi)$, and that $S_n(x)$ attains its minimum at $x = x_0$. Hence $\frac{d}{dx}[S_n(x)]_{x=x_0} = 0$. That is

$$S_n'(x_0) = \sum_{k=1}^{n} \cos((2k-1)x_0) = 0,$$

so that

$$\begin{aligned} 2\sin x_0 S_n'(x_0) &= \sum_{k=1}^{n} 2\cos((2k-1)x_0)\sin x_0 \\ &= \sum_{k=1}^{n} [\sin(2kx_0) - \sin((2k-2)x_0)] \\ &= \sin 2nx_0. \end{aligned}$$

Thus $S'_n(x_0) = \frac{\sin 2nx_0}{2\sin x_0} = 0$ implying $\sin 2nx_0 = 0$. Hence

$$x_0 \in \left\{\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots, \frac{(2n-1)}{2n}\right\}.$$

It is easily verified that at each of these values $S_n(x_0) > 0$, a contradiction. Hence $S_n(x) > 0$ for $x \in (0, \pi)$.

Editor's Note: Both solutions used the calculus. Does anyone have a more elementary solution?

We complete this number of the Corner with solutions by our readers to problems of the 1992 Dutch Mathematical Olympiad, Second Round given in the June 1995 number of the Corner [1995; 192–193].

1992 DUTCH MATHEMATICAL OLYMPIAD
Second Round
September 18, 1992

1. Four dice are thrown. What is the chance that the product of the numbers equals 36?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There are four different kinds of outcomes in which the product is 36: each of $\{1,1,6,6\}$ and $\{2,2,3,3\}$ can occur in $\frac{4!}{2!2!} = 6$ ways; $\{1,4,3,3\}$ can occur in $\frac{4!}{2!} = 12$ ways; and $\{1,2,3,6\}$ can occur in $4! = 24$ ways. Hence the probability that the product equals 36 is $\frac{48}{6^4} = \frac{1}{27}$.

2. In the fraction and its decimal notation (with period of length 4) every letter represents a digit. Different letters denote different digits. The numerator and denominator are mutually prime. Determine the value of the fraction:

$$\frac{ADA}{KOK} = .SNELSNELSNELSNEL\ldots$$

[*Note*. ADA KOK is a famous Dutch swimmer. She won gold in the 1968 Olympic Games in Mexico. SNEL is Dutch for FAST.]

Solution by the Editor.

Let $x = \frac{ADA}{KOK} = .\overline{SNEL}$. Then $10^4x = SNEL.SNELSNEL\ldots$ and $10^4x - x = SNEL$. So

$$x = \frac{SNEL}{9999} = \frac{SNEL}{11 \times 909} = \frac{SNEL}{33 \times 303} = \frac{SNEL}{99 \times 101}.$$

Taking $KOK = 909$ we obtain $SNEL = 11 \times ADA$, and $A = L$, which is impossible.

Taking $KOK = 303$, we obtain $SNEL = 33 \times ADA$, so $3A < 10$, as the product has four digits and $3A = L$. Because $S \neq L \neq 0$, $3 \leq 3A < 9$, giving $A = 1$ or $A = 2$. Now $A = 1$ gives $L = 3 = K$, which is impossible, so $A = 2$. This gives $L = 6$, and $D \geq 2$, so there is a carry. This gives $D \geq 4$, as $A = 2$, $K = 3$.

For $D = 4$, $\frac{ADA}{KOK} = .SNEL$ is $\frac{242}{303} = \overline{.7986}$, a solution.

For $D = 5$, $ADA = 252$ is not coprime to $KOK = 303$.

For $D = 6$, $SNEL = 8646$ and $N = L$.

For $D = 7$, $SNEL = 8976$ and $D = E$.

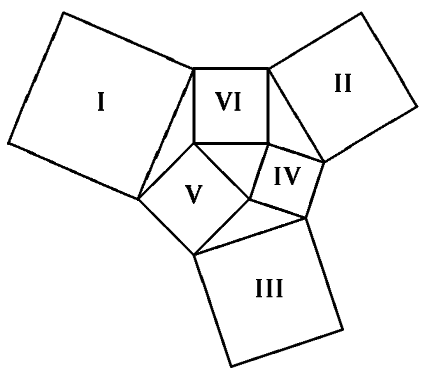
For $D = 8$, $SNEL = 9036$ and $O = N$.

For $D = 9$, $SNEL = 9636$ and $N = L$.

Taking $KOK = 101$ gives $SNEL = 99 \times ADA$ forcing $A = 1$ for $SNEL$ to have four digits, but then $A = K$.

Thus the only solution is $\frac{242}{303} = \overline{.7986}$.

3. The vertices of six squares coincide in such a way that they enclose triangles; see the picture. Prove that the sum of the areas of the three outer squares (I, II and III) equals three times the sum of the areas of the three inner squares (IV, V and VI).



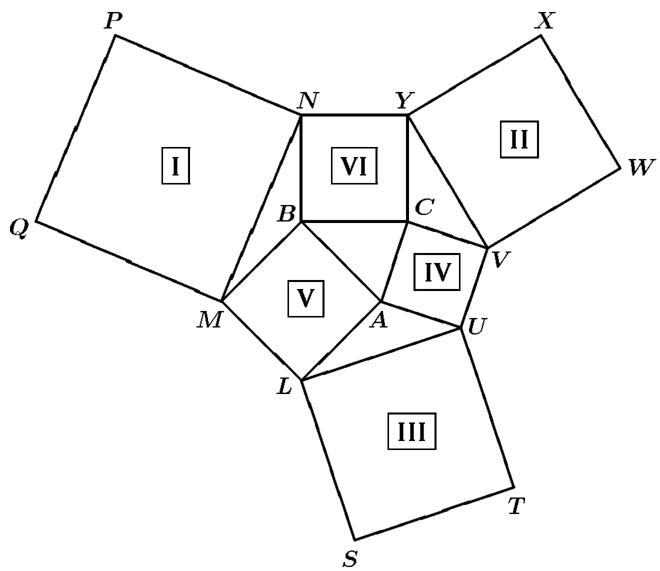
Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Vedula N. Murty, Andhra University, Visakhapatnam, India. We give Murty's solution.

Let the figure on the next page be labelled as shown:

Let $MN = x_1$, $LU = x_3$, $UY = x_2$, $AC = x_4$, $AB = x_5$, $BC = x_6$, $\angle MBN = \alpha$, $\angle LAU = \beta$, $\angle VCY = \gamma$, $\angle BAC = A$, $\angle ACB = C$ and $\angle ABC = B$.

Then we have $\alpha + \beta = \pi$, $\beta + A = \pi$, $\gamma + C = \pi$

$$\left.\begin{aligned} x_1^2 &= x_6^2 + x_5^2 - 2x_5x_6\cos\alpha \\ x_2^2 &= x_4^2 + x_6^2 - 2x_4x_6\cos\gamma \\ x_3^2 &= x_4^2 + x_5^2 - 2x_4x_5\cos\beta \end{aligned}\right\} \ldots \qquad (1)$$



$$\left.\begin{aligned} x_4^2 &= x_5^2 + x_6^2 - 2x_5x_6\cos B \\ x_4^2 &= x_4^2 + x_6^2 - 2x_4x_6\cos C \\ x_6^2 &= x_4^2 + x_5^2 - 2x_4x_6\cos A \end{aligned}\right\}\ldots \qquad (2)$$

From (2), we have

$$\begin{aligned} x_4^2 + x_5^2 + x_6^2 &= 2x_4x_5\cos A + 2x_5x_6\cos B + 2x_4x_6\cos C \\ &= -2x_4x_5\cos\beta - 2x_5x_6\cos\alpha - 2x_4x_6\cos\gamma \quad \ldots (3) \end{aligned}$$

From (1), we have

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 2(x_4^2 + x_5^2 + x_6^2) - 2x_5x_6\cos\alpha \\ &\quad -2x_4x_5\cos\beta - 2x_4x_6\cos\gamma \end{aligned}$$

using (3)

$$\begin{aligned} &= 2(x_4^2 + x_5^2 + x_6^2) + x_4^2 + x_5^2 + x_6^2 \\ &= 3(x_4^2 + x_5^2 + x_6^2). \end{aligned}$$

That is, Area of $(I + II + III) = 3$ Area of $(IV + V + VI)$.

4. For every positive integer n, $n?$ is defined as follows:

$$n? = \begin{cases} 1 & for\ n = 1 \\ \frac{n}{(n-1)?} & for\ n \geq 2 \end{cases}$$

Prove $\sqrt{1992} < 1992? < \frac{4}{3}\sqrt{1992}$.

Solutions by Vedula N. Murty, Andhra University, Visakhapatnam, India; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and his remark.

Using the more convenient notation $f(n)$ for $n?$, we show that in general

$$\sqrt{n+1} < f(n) < \frac{4}{3}\sqrt{n} \qquad (*)$$

for all **even** $n \geq 6$. In particular, for $n = 1992$, we would get $\sqrt{1993} < f(1992) < \frac{4}{3}\sqrt{1992}$.

First note that $f(n) = \frac{n}{f(n-1)} = \frac{n}{n-1} f(n-2)$ for all $n \geq 3$. If $N = 2k$ where $k \geq 2$, then multiplying $f(2q) = \frac{2q}{2q-1} f(2q-2)$ for $q = 2, 3, \ldots, k$, we get

$$\begin{aligned} f(2k) &= \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}\cdot f(2) \\ &= \left(\frac{2}{1}\right)\cdot\left(\frac{4}{3}\right)\cdot\left(\frac{6}{5}\right)\cdots\left(\frac{2k}{2k-1}\right) > \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\cdots\left(\frac{2k+1}{2k}\right). \end{aligned}$$

Hence

$$(f(2k))^2 > \frac{2\cdot 4\cdot 6\cdots 2k}{1\cdot 3\cdot 5\cdots(2k-1)}\cdot\frac{3\cdot 5\cdot 7\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} = 2k+1,$$

from which it follows that

$$f(n) = f(2k) > \sqrt{2k+1} = \sqrt{n+1}. \qquad (1)$$

On the other hand, for $k \geq 3$ we have

$$\begin{aligned} 2(2k) &= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{2k-2}{2k-1}\right)\cdot 2k \\ &< \left(\frac{2}{3}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\cdots\left(\frac{2k-1}{2k}\right) 2k. \end{aligned}$$

Hence

$$\begin{aligned} (f(2k))^2 &< \left(\frac{2}{3}\right)^2\cdot\frac{4\cdot 6\cdots(2k-2)}{5\cdot 7\cdots(2k-1)}\cdot\frac{5\cdot 7\cdots(2k-1)}{6\cdot 8\cdots 2k}\cdot(2k)^2 \\ &= \left(\frac{2}{3}\right)^2\cdot 4\cdot 2k, \end{aligned}$$

from which it follows that

$$f(n) = f(2k) < \frac{4}{3}\sqrt{2k} = \frac{4}{3}\sqrt{n}. \qquad (2)$$

The result follows from (1) and (2).

Remark: Using similar arguments, upper and lower bounds for $f(n)$ when n is odd can also be easily derived. In fact, if we set $P = \frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots2k}$ (usually denoted by $\frac{(2k-1)!!}{(2k)!!}$) then various upper and lower bounds for P abound in the literature; for example, it is known that

$$\frac{1}{2}\sqrt{\frac{5}{4k+1}} \le P \le \frac{1}{2}\sqrt{\frac{3}{2k+1}}$$

and

$$\frac{1}{\sqrt{(n+\frac{1}{2})\pi}} < P \le \frac{1}{\sqrt{n\pi}}.$$

(Compare, for example, §3.1.16 on p. 192 of *Analytic Inequalities* by D.S. Mitronović.)

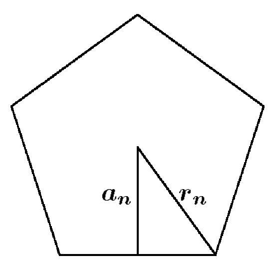
Clearly, each pair of these double inequalities would yield corresponding upper and lower bounds for the function $f(n)$ considered in the given problem.

5. We consider regular n-gons with a fixed circumference 4. We call the distance from the centre of such a n-gon to a vertex r_n and the distance from the centre to an edge a_n.

a) Determine a_4, r_4, a_8, r_8.

b) Give an appropriate interpretation for a_2 and r_2.

c) Prove: $a_{2n} = \frac{1}{2}(a_n + r_n)$ and $r_{2n} = \sqrt{a_{2n}r_n}$.



Let $u_0, u_1, u_2, u_3, \ldots$ be defined as follows:

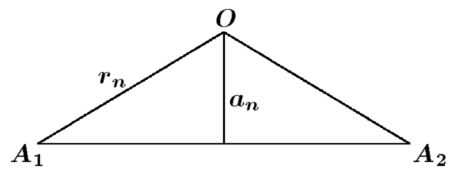
$$u_0 = 0,\ u_1 = 1;\ u_n = \frac{1}{2}(u_{n-2} + u_{n-1}) \text{ for } n \text{ even and}$$

$$u_n = \sqrt{u_{n-2} \cdot u_{n-1}} \text{ for } n \text{ odd.}$$

d) Determine: $\lim_{n\to\infty} u_n$.

Solution by Vedula N. Murty, Andhra University, Visakhapatnam, India.

Let O be the centre of the regular n-gon. Let A_1A_2 denote one side of the regular n-gon



Then we have $\angle A_1OA_2 = \frac{2\pi}{n}$, $\angle OA_1A_2 = \angle OA_2A_1 = \frac{\pi}{2} - \frac{\pi}{n}$. Thus

$$\begin{aligned} |\overrightarrow{A_1A_2}| &= \sqrt{r_n^2 + r_n^2 - 2r_n^2 \cos \frac{2\pi}{n}} \\ &= \sqrt{2r_n^2(1 - \cos \frac{2\pi}{n})} \\ &= \sqrt{4r_n^2 \sin^2 \frac{\pi}{n}} = 2r_n \sin \frac{\pi}{n}. \end{aligned}$$

The circumference of the regular n-gon is $2nr_n \sin \frac{\pi}{n} = 4$ whence

$$r_n = \frac{2}{n \sin \frac{\pi}{n}},$$

$$a_n = r_n \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = r_n \cos \frac{\pi}{n} = \frac{2}{n} \cot \frac{\pi}{n}.$$

In particular

$$r_4 = \frac{1}{2} \frac{1}{\sin \frac{\pi}{4}} = \frac{\sqrt{2}}{2}, \qquad a_4 = \frac{2}{4} \cot \frac{\pi}{4} = \frac{1}{2},$$

$$r_8 = \frac{2}{8 \sin \frac{\pi}{8}} = \frac{1}{4 \sin \frac{\pi}{8}}.$$

Now, $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = 1 - 2\sin^2 \frac{\pi}{8}$ gives

$$\sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}},$$

so

$$r_8 = \frac{1}{4} \frac{2}{\sqrt{2 - \sqrt{2}}} = \frac{1}{2} \cdot \frac{1}{\sqrt{2 - \sqrt{2}}},$$

and

$$a_8 = r_8 \cos \frac{\pi}{8} = \frac{1}{4}\sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}} = \frac{1}{4} \frac{1}{2 - \sqrt{2}}\sqrt{2},$$

since $\cos \frac{\pi}{4} = 2\cos^2 \frac{\pi}{8} - 1$.

For (b), $r_2 = 1$, $a_2 = 0$ as the 2-gon is a straight line with O lying at the middle of A_1 and A_2.

For (c), we have

$$\begin{aligned} a_n + r_n &= r_n\left(1+\cos\frac{\pi}{n}\right) = 2r_n\cos^2\frac{\pi}{2n} \\ &= \frac{4}{n\sin\frac{\pi}{n}}\cos^2\frac{\pi}{2n} \\ &= \frac{4}{2n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\cos^2\frac{\pi}{2n} = \frac{2}{n}\cot\frac{\pi}{2n}. \end{aligned}$$

Thus $\frac{1}{2}(a_n + r_n) = \frac{1}{n}\cot(\frac{\pi}{2n}) = a_{2n}$, and

$$a_{2n}r_n = \frac{1}{n}\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\cdot\frac{2}{n\sin\frac{\pi}{n}} = \frac{1}{n^2}\frac{\cos\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{1}{n^2\sin^2\frac{\pi}{2n}},$$

so $\sqrt{a_{2n}r_n} = \frac{1}{n\sin\frac{\pi}{2n}} = r_{2n}$.

For (d), note $u_0 = 0$, $u_1 = 1$, and $u_2 = \frac{1}{2}$. For $n \geq 2$ we have that u_n is either the arithmetic or geometric mean of u_{n-1} and u_{n-2} and in either case lies between them. It is also easy to show by induction that $u_0, u_2, u_4, \ldots$ form an increasing sequence, and $u_1, u_3, u_5, \ldots$ form a decreasing sequence with $u_{2l} \leq u_{2s+1}$ for all $l, s \geq 0$. Let $\lim_{k\to\infty} u_{2k} = P$ and $\lim_{k\to\infty} u_{2k+1} = I$. Then $P \leq I$. We also have from $u_{2n} = \frac{1}{2}(u_{2n-1} + u_{2n-2})$ that $P = \frac{1}{2}(I + P)$ so that $I = P$ and $\lim_{n\to\infty} u_n$ exists. Let $\lim_{n\to\infty} u_n = L$.

With $a_2 = 0$ and $r_2 = 1$, let $\overline{u}_{2k} = a_{2^{k+1}}$ and $\overline{u}_{2k+1} = r_{2^{k+1}}$, for $k = 0, 1, 2, \ldots$. From (c), $\overline{u}_0 = a_{2^1} = a_2 = 0$ and $\overline{u}_1 = r_{2^1} = r_2 = 1$. Also for $n = 2k+2$, $\overline{u}_{2k+2} = a_{2^{k+1+1}} = a_{2.2^{k+1}} = \frac{1}{2}(a_{2^{k+1}} + b_{2^{k+1}}) = \frac{1}{2}(\overline{u}_{2k} + \overline{u}_{2k+1})$; that is $\overline{u}_n = \frac{1}{2}(\overline{u}_{n-2} + \overline{u}_{n-1})$ and for $n = 2k+3$

$$\begin{aligned} \overline{u}_{2k+3} &= \overline{u}_{2(k+1)+1} = r_{2^{k+1+1}} = r_{2(2^{k+1})} \\ &= \sqrt{a_{2(2^{k+1})}\cdot r_{2^{k+1}}} = \sqrt{a_{2^{k+1+1}}\cdot r_{2^{k+1}}} \\ &= \sqrt{\overline{u}_{2(k+1)}\cdot\overline{u}_{2k+1}} \end{aligned}$$

so $\overline{u}_n = \sqrt{\overline{u}_{n-1}\cdot\overline{u}_{n-2}}$. Thus u_n and $\overline{u}_n$ satisfy the same recurrence and it follows that $L = \lim_{k\to\infty} a_{2^{k+1}} = \lim_{k\to\infty} r_{2^{k+1}}$. Now, from the solution to (c),

$$r_n = \frac{2}{n\sin\frac{\pi}{n}} = \frac{2}{\pi}\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}},$$

so $\lim_{n\to\infty} r_n = \frac{2}{\pi}$ since $\frac{\pi}{n} \to 0$. Therefore $\lim_{n\to\infty} u_n = \frac{2}{\pi}$.

That completes the column for this issue. Olympiad season is approaching. Send me your contests and nice solutions.