SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2101. [1996: 33] *Proposed by Ji Chen, Ningbo University, China.*

Let a, b, c be the sides and A, B, C the angles of a triangle. Prove that for any $k \le 1$,

$$\sum \frac{a^k}{A} \ge \frac{3}{\pi} \sum a^k,$$

where the sums are cyclic. [The case $k = 1$ is known; see item 4.11, page 170 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*.]

I. *Solution by Hoe Teck Wee, Singapore.*

Let $f(x) = \dfrac{\sin^k x}{x}$ for $0 < x < \pi$. Then, for $0 < k \le 1$, we have

$$f'(x) = \frac{(kx\cos x - \sin x)\sin^{k-1} x}{x^2}.$$

For $x \ge \pi/2$, we have $\cos x \le 0$, so that $kx\cos x - \sin x \le 0$.

For $x < \pi/2$, we have $\cos x > 0$ and $\tan x > x \ge kx$, so that $kx\cos x - \sin x \le 0$.

Therefore $f'(x) = \dfrac{(kx\cos x - \sin x)\sin^{k-1} x}{x^2} \le 0$.

Without loss of generality, we may assume that $A \le B \le C$. For $0 < k \le 1$, we have that $f(x)$ is a non-increasing function, so that $f(A) \ge f(B) \ge f(C)$. Thus, by Tchebyshev's inequality, we have

$$\left(\frac{\sin^k A}{A} + \frac{\sin^k B}{B} + \frac{\sin^k C}{C}\right)(A+B+C) \ge 3\left(\sin^k A + \sin^k B + \sin^k C\right).$$

By the Sine Rule, we have $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$, where R is the circumradius of $\triangle ABC$. Multiply the inequality by $(2R)^k$ and substitute $A + B + C = \pi$ to get

$$\sum \frac{a^k}{A} \ \ge\ \frac{3}{\pi} \sum a^k. \qquad (1)$$

For $k \le 0$, we have

$$a \le b \le c \Longrightarrow a^k \ge b^k \ge c^k \Longrightarrow \frac{a^k}{A} \ge \frac{b^k}{B} \ge \frac{c^k}{C}.$$

Thus, by using Tchebyshev's inequality again, (1) holds for $k \le 0$.

In conclusion, (1) holds for any $k \leq 1$.

II. *Solution by Kee-Wai Lau, Hong Kong.* (Edited)

We let, without loss of generality, $a \geq b \geq c$, so that $A \geq B \geq C$. Next, put $f(k) = \left(a^k/A + b^k/B + c^k/C\right)\left(a^k + b^k + c^k\right)^{-1}$, then

$$f'(k) = -\sum \frac{a^k b^k (A-B)(\log a - \log b)}{AB} \left(\sum a^k\right)^{-2} \leq 0,$$

where the sums are cyclic. Since we are given that $f(1) \geq 3/n$, it follows that $f(k) \geq 3/n$ for $k \leq 1$.

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. One incorrect solution was received.

Janous establishes the following generalization.

Let k and ℓ be real numbers. Then:

1.
$$\sum \frac{a^k}{A^\ell} \geq \left(\frac{3}{\pi}\right)^\ell \sum a^k,$$

for the cases

- $0 \leq k \leq \ell \leq 1$,
- $k \leq 0 \leq \ell$,
- $k \geq 0$ and $\ell \leq -1$.

2.
$$\sum \frac{a^k}{A^\ell} \leq \left(\frac{3}{\pi}\right)^\ell \sum a^k,$$

where

$$k \leq 0 \quad \text{and} \quad -1 \leq \ell \leq 0.$$

2102. [1996: 33] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. Let P and Q be the feet of the perpendiculars from A to BI and CI respectively. Prove that

$$\frac{AP}{BI} + \frac{AQ}{CI} = \cot\frac{A}{2}.$$

Solution by Panos E. Tsaoussoglou, Athens, Greece, and by eight others!

In $\triangle APB$, we have $\sin(B/2) = \dfrac{AP}{AB}$,
in $\triangle AQC$, we have $\sin(C/2) = \dfrac{AQ}{AC}$,
in $\triangle ABI$, we have $\dfrac{BI}{\sin(A/2)} = \dfrac{AB}{\cos C/2}$,
in $\triangle ACI$. we have $\dfrac{CI}{\sin(A/2)} = \dfrac{AC}{\cos B/2}$,
so that

$$\begin{aligned}\frac{AP}{BI} + \frac{AQ}{CI} &= \frac{\sin(B/2)\ \cos(C/2)}{\sin(A/2)} + \frac{\sin C/2\ \cos B/2}{\sin A/2} \\ &= \frac{\sin(B/2\,+\,C/2)}{\sin(A/2)} = \frac{\cos(A/2)}{\sin(A/2)} = \cot(A/2).\end{aligned}$$

Editor's comment. Our featured solution is based on the property: $\angle AIB = \pi - A/2 - B/2 = \pi/2 + C/2$, so that $\sin(\angle AIB) = \cos(C/2)$. Other solvers used the equally efficient relations: $BI = r/\sin(B/2)$, or $BI = 4R\sin(A/2)\sin(C/2)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (two solutions); FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; MELITIS D. VASILIOU. Elefsis, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

2103. [1996: 33] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle. Let D be the point on side BC produced beyond B such that $BD = BA$, and let M be the mid-point of AC. The bisector of $\angle ABC$ meets DM at P. Prove that $\angle BAP = \angle ACB$.

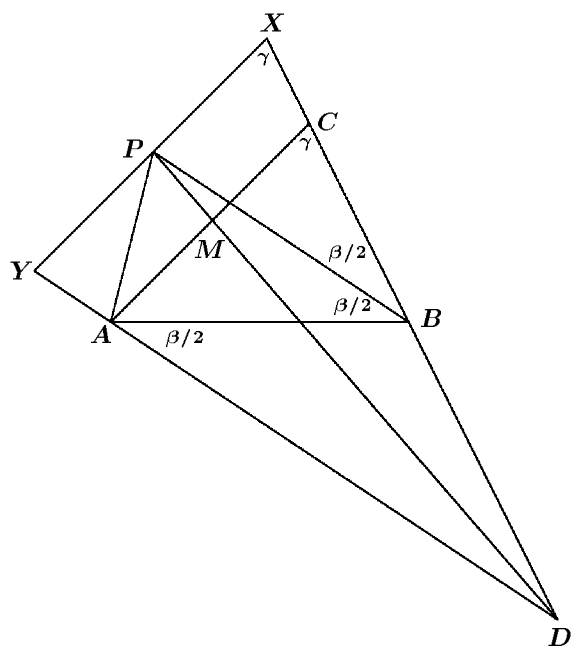
Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.

Let PX be parallel to AC with X lying on the line BC. Let Y be the intersection of PX with AD. P is the midpoint of XY because M is the mid-point of AC.

Then B is the mid-point of DX [PB is parallel to AD since $2\angle DAB = \angle DAB + \angle BDA = \angle ABC = 2\angle PBA$].

Hence $BX = BD = AB$. Triangle BPA is congruent to triangle BPX [$PB = PB$; $AB = XB$; $\angle ABP = \angle XBP$.]

Therefore, $\angle BAP = \angle BXP = \angle BCA$ [$PX \| AC$].



Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; TIM CROSS, King Edward's School, Birmingham, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO

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2104. [1996: 34] *Proposed by K.R.S. Sastry, Dodballapur, India.*
In how many ways can 111 be written as a sum of three integers in geometric progression?

Solution by Zun Shan, Nanjing Normal University, Nanjing, China.

The answer is seventeen or sixteen depending on whether we allow the common ratio of the G.P. (geometric progression) to be zero or not.

Suppose $111 = a + ar + ar^2$ where a is an integer and r is a rational number. If $r = 0$, then we get the trivial solution

$$111 = 111 + 0 + 0. \tag{1}$$

Suppose $r = \frac{n}{m} \neq 0$ where m and n are nonzero integers. Without loss of generality, we may also assume that $m > 0$ and $(m, n) = 1$. Since the reverse of the G.P. a, ar, ar^2 is another G.P. ar^2, ar, a, we may also assume that $|r| \geq 1$ and so $0 < m \leq |n|$.

From $a(1 + r + r^2) = 111$ we get $a(m^2 + mn + n^2) = 111m^2$. Since clearly $(m^2 + mn + n^2, m^2) = 1$ we have $m^2|a$. Letting $a = km^2$ where k is an integer we then get $k(m^2 + mn + n^2) = 111$ which implies $k|111$. Since $m^2 + mn + n^2 > 0$ and $111 = 3 \times 37$, we have $k = 1, 3, 37$, or 111. Note that $m^2 + mn + n^2 = m^2 + |n|(\pm m + |n|) \geq m^2$.

Case [1] If $k = 1$, then $m^2 + mn + n^2 = 111 \Longrightarrow m^2 \leq 111 \Longrightarrow m \leq 10$. When $m = 1$, $a = 1$ and from $n^2 + n = 110$ we get $n = 10, -11$. Thus $r = 10$, -11 and we obtain the solutions:

$$111 = 1 + 10 + 100 = 1 - 11 + 121. \tag{2}$$

For $2 \leq m \leq 9$ it is easily checked that the resulting quadratic equation in n has no integer solutions.

When $m = 10$, $a = 100$ and from $n^2 + 10n = 11$ we get $n = 1, -11$. Since $m \leq |n|$, $n = -11$ and $r = -\frac{11}{10}$ yielding the solution:

$$111 = 100 - 110 + 121. \tag{3}$$

Case [2] If $k = 3$, then $m^2 + mn + n^2 = 37 \Longrightarrow m^2 \leq 37 \Longrightarrow m \leq 6$. Quick checkings reveal that there are no solutions for $m = 1, 2, 5, 6$.

When $m = 3$, $a = 27$ and from $n^2 + 3n = 28$ we get $n = 4, -7$. Thus $r = \frac{4}{3}, -\frac{7}{3}$ yielding the solutions:

$$111 = 27 + 36 + 48 = 27 - 63 + 147. \tag{4}$$

When $m = 4$, $a = 48$ and from $n^2 + 4n = 21$ we get $n = 3, -7$.

Since $m \le |n|$, $n = -7$ and $r = -\frac{7}{4}$ yielding the solution:

$$111 = 48 - 84 + 147. \qquad (5)$$

Case [3] If $k = 37$, then $m^2 + mn + n^2 = 3 \Longrightarrow m^2 \le 3 \Longrightarrow m = 1 \Longrightarrow a = 37$ and from $n^2 + n = 2$ we get $n = 1, -2$. Thus $r = 1$ or -2 yielding the solutions:

$$111 = 37 + 37 + 37 = 37 - 74 + 148. \qquad (6)$$

Case [4] If $k = 111$, then $m^2 + mn + n^2 = 1 \Longrightarrow m^2 \le 1 \Longrightarrow m = 1 \Longrightarrow a = 111$, and from $n^2 + n = 0$ we get $n = -1$ as $n \ne 0$. Thus $r = -1$ and we get the solution

$$111 = 111 - 111 + 111. \qquad (7)$$

Reversing the summand in (2) – (7) and noting that two of them are "symmetric", we obtain seventeen solutions in all:

$$\begin{array}{rlllll} 111 & = 111 + 0 + 0 & = 1 + 10 + 100 & = 100 + 10 + 1 \\ & = 1 - 11 + 121 & = 121 - 11 + 1 & = 100 - 110 + 121 \\ & = 121 - 110 + 100 & = 27 + 36 + 48 & = 48 + 36 + 27 \\ & = 27 - 63 + 147 & = 147 - 63 + 27 & = 48 - 84 + 147 \\ & = 147 - 84 + 48 & = 37 + 37 + 37 & = 37 - 74 + 148 \\ & = 148 - 74 + 37 & = 111 - 111 + 111. & \end{array}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain, (who assumed $r \ne 0$ and found sixteen solutions); F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; (both of them found all seventeen solutions). There were also two incomplete and twenty-three incorrect solutions submitted!

Among these twenty-three, thirteen submissions claimed six solutions; six submissions claimed five solutions; two submissions claimed six or nine solutions; one submission claimed two solutions, and one submission claimed one solution only. Most of the errors were the result of assuming by mistake that $a(1 + r + r^2) = 111 \Longrightarrow 1 + r + r^2$ must be an integer.

2105. [1996: 34] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK*

Find all values of λ for which the inequality

$$2(x^3 + y^3 + z^3) + 3(1 + 3\lambda)xyz \ge (1 + \lambda)(x + y + z)(yz + zx + xy)$$

holds for all positive real numbers x, y, z.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

On setting $x = y = 1$ and $z = 0$, we obtain $4 \geq (1 + \lambda)2$ and thus find that λ must be ≤ 1. We now show that the inequality holds for all $\lambda \leq 1$.

First, if $\lambda = 1$, the inequality reduces to

$$x^3 + y^3 + z^3 + 6xyz \geq (x + y + z)(yz + zx + xy),$$

which is equivalent to the special case $n = 1$ of the known Schur inequality

$$x^n(x - y)(x - z) + y^n(y - z)(y - x) + z^n(z - x)(z - y) \geq 0,$$

true for all real n, and which has come up many times in this journal. The rest will follow by showing that for all $\lambda < 1$,

$$\begin{aligned} &(1 + \lambda)(x + y + z)(yz + zx + xy) - 3(1 + 3\lambda)xyz \\ \leq\ &(1 + 1)(x + y + z)(yz + zx + xy) - 3(1 + 3)xyz. \end{aligned} \tag{1}$$

[*Editor's note*: rewrite the original inequality as

$$2(x^3 + y^3 + z^3) \geq (1 + \lambda)(x + y + z)(yz + zx + xy) - 3(1 + 3\lambda)xyz;$$

then (1) says that the right hand side is largest when $\lambda = 1$, so doing the case $\lambda = 1$ would be enough.] But (1) can be written

$$(1 - \lambda)[(x + y + z)(yz + zx + xy) - 9xyz] \geq 0$$

which [after cancelling the positive factor $1 - \lambda$] is a known elementary inequality, equivalent to Cauchy's inequality

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9.$$

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and HOE TECK WEE, Singapore. There were three incorrect solutions sent in.

2106. [1996: 34] *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A quadrilateral has sides a, b, c, d (in that order) and area F. Prove that

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F.$$

When does equality hold?

Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.

Let $ABCD$ be the quadrilateral with $AB = a$, $BC = b$, $CD = c$, and $DA = d$. We can assume, without loss of generality, that $AC = 1$. Therefore, we can locate the quadrilateral in a system of Cartesian coordinates where

$$A = (0,0), \quad B = (p,q), \quad C = (1,0), \quad D = (r,s).$$

We assume that $ABCD$ is simple so that its area is well-defined. If $ABCD$ is not convex we can make it convex and keep the side lengths the same while increasing the area. This means that we will be done if we can show that the result is true for convex quadrilaterals. It's also clear from this that if the result is true for convex quadrilaterals, then equality cannot hold for non-convex quadrilaterals. Therefore, assume $q < 0$ and $s > 0$. Now note that

$$\begin{aligned} 2a^2 + 5b^2 &= 2(p^2+q^2) + 5((p-1)^2+q^2) \\ &= 7p^2 - 10p + 5 + 7q^2 \\ &= 7\left(p - \tfrac{5}{7}\right)^2 - \tfrac{25}{7} + 5 + 7q^2 \\ 2a^2 + 5b^2 &\geq 7q^2 + \tfrac{10}{7}, \end{aligned} \qquad (2)$$

and

$$\begin{aligned} 8c^2 - d^2 &= 8((r-1)^2+s^2) - (r^2+s^2) \\ &= 7r^2 - 16r + 8 + 7s^2 \\ &= 7\left(r - \tfrac{8}{7}\right)^2 - \tfrac{64}{7} + 8 + 7s^2 \\ 8c^2 - d^2 &\geq 7s^2 - \tfrac{8}{7}. \end{aligned} \qquad (3)$$

Combining (2) and (3), we get

$$\begin{aligned} & 2a^2 + 5b^2 + 8c^2 - d^2 \\ &\geq 7q^2 + 7s^2 + \tfrac{2}{7} \\ &= \left(7q^2 + \tfrac{1}{7}\right) + \left(7s^2 + \tfrac{1}{7}\right) \\ &= \left(7\left(|q| - \tfrac{1}{7}\right)^2 + 2|q|\right) + \left(7\left(|s| - \tfrac{1}{7}\right)^2 + 2|s|\right) \\ &\geq 2(|q| + |s|) \qquad (4) \\ &= 4(\overline{ABC}) + 4(\overline{CDA}). \end{aligned}$$

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F,$$

as we wished to prove (where $\overline{ABC}$ and $\overline{CDA}$ refer to the areas of the two triangles ABC and CDA respectively). For equality to hold (when $A = (0,0)$ and $C = (1,0)$), it must hold in steps (2), (3), and (4). Therefore $p = \frac{5}{7}$, $r = \frac{8}{7}$, $q = -\frac{1}{7}$, and $s = \frac{1}{7}$. Thus, in general, equality holds if and only if $ABCD$ is directly similar to quadrilateral $A_0B_0C_0D_0$, where

$$A_0 = (0,0), \quad B_0 = \left(\tfrac{5}{7}, -\tfrac{1}{7}\right), \quad C_0 = (1,0), \quad D_0 = \left(\tfrac{8}{7}, \tfrac{1}{7}\right).$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer.

KONEČNÝ makes the observation that the quadrilateral is cyclic and that $\angle ABD = 135°$. The proposer makes the early observation that the maximum area for a quadrilateral with fixed sides occurs when it is cyclic and uses properties of cyclic quadrilaterals in the proof. He also generalizes the result to an inequality which JANOUS uses in his proof and which appears in the Addenda to the Monograph "Recent Advances in Geometric Inequalities" *by D. S. Mitrinović et al. in I. Journal of Ningbo University 4, No. 2 (Dec. 1991), 79–145.*

2107. [1996: 34] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC is not isosceles nor equilateral, and has sides a, b, c. D_1 and E_1 are points of BA and CA or their productions so that $BD_1 = CE_1 = a$. D_2 and E_2 are points of CB and AB or their productions so that $CD_2 = AE_2 = b$. Show that $D_1E_1 \parallel D_2E_2$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let S be the intersection of AB and D_2E_1.

[*Editor's comment by Chris Fisher.* Even though there seem to be two choices for each D_i and E_i, no solver had any trouble choosing the positions that make the result correct; furthermore, it must have been "obvious" to evey-one but me that AB is not parallel to D_2E_1, so that S exists. Alas, perhaps I need stronger glasses.]

Then CS is the bisector of $\angle ACB$, since $CE_1 = CB$ and $CA = CD_2$. Therefore

$$\frac{D_1S}{SE_2} = \frac{BD_1 - BS}{AE_2 - AS} = a = BSb - AS = \frac{a}{b},$$

since $\dfrac{BS}{AS} = \dfrac{a}{b}$. It then follows that $D_1E_1 \| D_2E_2$, since

$$\frac{E_1S}{SD_2} = \frac{CE_1}{CD_2} = \frac{a}{b} = \frac{D_1S}{SE_2}.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; JOEL SCHLOSBERG, student, Hunter

College High School, New York NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; MELITIS D. VASILIOU. Elefsis, Greece; and the proposer.

Janous adds the observation that D_1E_1 and D_2E_2 are not only parallel, but their lengths are in the ratios $a : b$ (as is clear from the featured solution).

2108. [1996: 34] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Prove that

$$\frac{a+b+c}{3} \leq \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}}\,,$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria, (modified slightly by the editor).

By the arithmetic-geometric mean inequality we have

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 6\sqrt[6]{a^6b^6c^6} = 6abc,$$

which implies

$$\begin{aligned} &9(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc) \\ &\quad\geq\ 8(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc), \end{aligned}$$

or

$$\begin{aligned} &9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca) \\ &\quad=\ 4(a+b+c)(a(b+c)+b(c+a)+c(a+b)). \end{aligned}$$

Using the arithmetic-geometric mean inequality again, we then have

$$\begin{aligned} &\frac{3}{4}(a+b)(b+c)(c+a) \\ &\quad\geq\ (a+b+c)\cdot\frac{a(b+c)+b(c+a)+c(a+b)}{3} \\ &\quad\geq\ (a+b+c)\sqrt{abc(a+b)(b+c)(c+a)} \qquad (1) \end{aligned}$$

From (1) it follows immediately that

$$\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}} \geq \frac{a+b+c}{3}.$$

Clearly, equality holds if $a = b = c$. [Ed. In fact, if equality holds, then from (1) we have $a(b+c) = b(c+a) = c(a+b)$. The first equality implies

$a = b$ and the second one implies $b = c$. Thus, equality holds in the given inequality if and only if $a = b = c$. This was observed by about half of the solvers.]

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous commented that upon the transformation

$$a \to \frac{1}{a},\ b \to \frac{1}{b},\ c \to \frac{1}{c},$$

the given inequality can be shown to be equivalent to

$$\sqrt{\frac{ab+bc+ca}{3}} \le \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}$$

which is known in the literature as Carlson's inequality (cf. eg. P.S. Bullen, D.S. Mitrinovic and P.M. Vasic, "Means and Their Inequalities", *Dordrechf, 1988. An anonymous reader commented that in this form, the inequality was problem 3 of the 1992 Austrian-Polish Mathematics Competition and has appeared in Crux before (see* [1994:97; 1995:336-337]). *Several solvers showed that the given inequality is equivalent to various other trigonometric inequalities involving a triangle XYZ, for example*

$$\cos\left(\frac{X}{2}\right)\cos\left(\frac{Y}{2}\right)\cos\left(\frac{Z}{2}\right) \le \frac{3\sqrt{3}}{8}$$

or

$$\sin X + \sin Y + \sin Z \le \frac{3\sqrt{3}}{2},$$

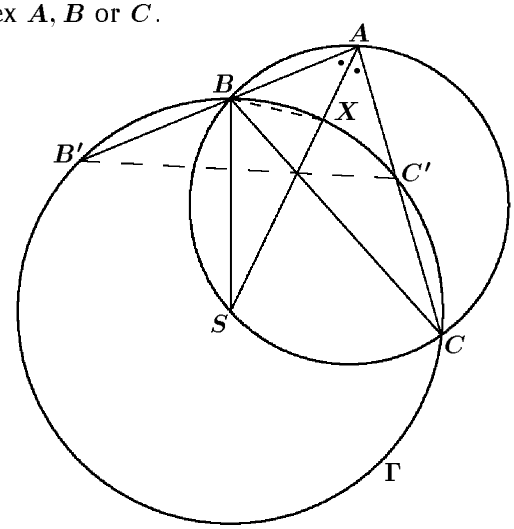
etc. These inequalities can be found in "Geometric Inequalities" *by O. Bottema et al.*

2109. [1996: 34] *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given a triangle and a circle passing through two of the vertices of the triangle and also through the incentre of the triangle. (The incentre and the centre of the circle are not given.) Construct, using only an unmarked ruler, the incentre.

Solution by P. Penning, Delft, the Netherlands.

Let the triangle be ABC, and Γ the circle passing through B, C and the incentre. The angles of the triangle are denoted by the symbol for the corresponding vertex A, B or C.



ANALYSIS:

Let point S be the intersection of the **circumcircle** and the angular bisector through A. The arcs SB and SC of the circumcircle are now equal and so are the chords SB and SC. Introduce X on AS such that $SX = SB = SC$.

$\angle BSA = \angle BCA = C$;

$\triangle XBS$ is isosceles with $SX = SB$, so $\angle XBS = 90^\circ - C/2$.

$\angle CBS = \angle CAS = A/2$; $\angle XBC = 90^\circ - C/2 - A/2 = B/2$.

So BX is the angular bisector at vertex B, and X must be the incentre. The circle Γ apparently has the point S as centre. [*It does, see Roger A. Johnson,* Modern Geometry, *292*].

[*Editor's note: If either AB or AC is tangent to Γ, then they both are and $AB = AC$. Suppose AB is tangent to Γ. Then $\angle SBA = \frac{\pi}{2}$, so $\angle BSA + \angle SAB = \frac{\pi}{2}$. Since $SC = SB$, $\angle BCS = \angle SBC = \angle SAC = \angle SAB$. Therefore, $\angle ACS = \angle ACB + \angle BCS = \angle BSA + \angle SAB = \frac{\pi}{2}$ and AC is tangent to Γ. In addition, tangents to a circle from an exterior point are equal, so $AB = AC$.*]

So if $AB \neq AC$, neither line is tangent to Γ. Let B' and C' be the other intersections of AB, respectively AC, with Γ. There is mirror-symmetry with respect to the line AS : Γ remains Γ; AB reflects into AC and AC reflects into AB; B and C' are mirror-images and so are C and B'. The side BC becomes $B'C'$; as a consequence they must intersect on the mirror-line AS.

CONSTRUCTION:

Find the other two intersections, B' and C', of AB and AC with the circle Γ. The intersection of BC and $B'C'$ is M. The incentre is the intersection of AM and Γ.

COMMENT:

The construction fails if ABC is isosceles, with $AB = AC$. In that case Γ touches both AB and AC in B and C respectively. M is now the midpoint of BC, but that cannot be found with unmarked ruler.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer.

2110. [1996: 35] *Proposed by Jordi Dou, Barcelona, Spain.*

Let S be the curved *Reuleaux* triangle whose sides AB, BC and CA are arcs of unit circles centred at C, A and B respectively. Choose at random (and uniformly) a point M in the interior and let $C(M)$ be a chord of S for which M is the midpoint. Find the length ℓ such that the probability that $C(M) > \ell$ is $1/2$.

Solution by the proposer.

Let Σ be the locus of the mid-point M of segments of constant length σ, whose ends S_1 and S_2 move on the boundary of S. For the points M inside Σ the chords bisected by M are greater than σ.

($\star$) The area contained between S and Σ is $\frac{\pi}{4}\sigma^2$. It is sufficient to show that

$$\frac{\pi}{4}\ell^2 = \frac{[S]}{2}.$$

Since $[S] = \frac{\pi}{2} - \frac{2\sqrt{3}}{4}$, we will have

$$\ell = \left(\frac{\pi - \sqrt{3}}{\pi}\right)^{\frac{1}{2}} \simeq 0.67.$$

Brief proof of the assertion ($\star$) (**Special Case of Holditch's Theorem**)

The ends S_1, S_2 of the chord of constant length move along the contour of the closed curve S. The mid-point M describes the curve Σ.

Let $S_1 = (x_1, y_1)$, $S_2 = (x_2, y_2)$, and $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. Suppose that x_1, y_1, x_2, y_2 are functions of t such that for $t_0 \le t \le t_1$, we have S_1, S_2 describing S.

$$\begin{aligned}[S] &= \int_{t_0}^{t_1} y_1\,dx_1 &= \int_{t_0}^{t_1} y_2\,dx_2 = \int_{t_0}^{t_1} \frac{1}{2}(y_1\,dx_1 + y_2\,dx_2) \\ [\Sigma] &= \int_{t_0}^{t_1} y\,dx &= \int_{t_0}^{t_1} \frac{1}{4}(y_1 + y_2)(dx_1 + dx_2).\end{aligned}$$

Then

$$[S] - [\Sigma] = \frac{1}{4}\int_{t_0}^{t_1} (y_2 - y_1)(dx_2 - dx_1).$$

Substitute $X = x_2 - x_1$, $Y = y_2 - y_1$, giving

$$[S] - [\Sigma] = \frac{1}{4}\int_{t_0}^{t_1} Y\,dX = \frac{\pi}{4}\sigma^2,$$

since (X, Y) describes a circle of radius σ.

2111. [1996: 35] *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Does there exist a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) satisfying the three conditions:

(i) $f(1996) = 1$;

(ii) for all primes p, every prime occurs in the sequence

$f(p)$, $f(2p)$, $f(3p)$, $\ldots$, $f(kp)$, $\ldots$ infinitely often; and

(iii) $f(f(n)) = 1$ for all $n \in \mathbb{N}$?

I. *Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Yes, a function does exist that satisfies the three conditions. It is:

$$f(x) = \begin{cases} p_i & \text{if condition } * \text{ holds,} \\ 1 & \text{otherwise,} \end{cases}$$

where condition $*$ is: if the prime factorization of x is $x = \prod p_i^{e_i}$, there exists a power e_i such that $e_i > 2$ and $e_i > e_j$ for all $j \neq i$.

For example, 109850 has condition $*$, since $109850 = 2 \times 5^2 \times 13^3$ and the power of 13 is bigger than 2 and bigger than all other powers in the factorization; thus $f(109850) = 13$.

Now

- f satisfies condition (i) since $f(1996) = f(2^2 \times 499) = 1$;
- f satisfies condition (ii) because for any two primes p and q, $f(x_i) = q$ for every $x_i = q^i p$, $i = 3, 4, \ldots$;
- f satisfies condition (iii) since for all n either $f(n) = 1$ or $f(n) = p_i$ for some prime p_i, and in either case $f(f(n)) = 1$.

II. *Solution by Chris Wildhagen, Rotterdam, the Netherlands.*

For each $n \in \mathbb{N}$ let q_n be the nth prime and $b(n)$ be the number of 1's in the binary representation of n. Define $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$f(m) = \begin{cases} q_{b(n)} & \text{if } m = p^n \text{ with } p \text{ prime and } n \geq 2, \\ 1 & \text{else.} \end{cases}$$

Clearly f satisfies the three given conditions.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; DAVID E. MANES, State University of New York, Oneonta, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposer. There were three incorrect solutions sent in.

Most solvers gave a variation of Solution I.

2112. [1996: 35] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find a four-digit base-ten number $abcd$ (with $a \neq 0$) which is equal to $a^a + b^b + c^c + d^d$.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario (modified slightly by the editor).

We first stipulate that $0^0 = 1$. Let $m = abcd$, $s = a^a + b^b + c^c + d^d$ and assume that $m = s$. Clearly, $10^3 \leq m < 10^4$. If $x \geq 6$ for any $x \in \{a, b, c, d\}$ then $s \geq 6^6 > 10^4$ which is a contradiction. So $a, b, c, d \leq 5$.

If $x < 5$ for all $x \in \{a, b, c, d\}$, then $s \leq 4 \times 4^4 = 1024$ and furthermore $a = b = c = d = 4$ is the only combination for which $s \geq 10^3$. However, in this case, $s = 1024 \neq 4444 = m$. Hence $x = 5$ for some $x \in \{a, b, c, d\}$. We cannot have more than one 5 or else $s \geq 2 \times 5^5 = 6250$ would imply that some digit of m is at least 6. Hence, we have exactly one 5.

Since $s > 5^5 = 3125$ and $s \leq 5^5 + 3 \times 4^4 = 3893 < 4000$, we must have $a = 3$. Thus, $s = 5^5 + 3^3 + x^x + y^y = 3152 + x^x + y^y$ where $x, y \in \{a, b, c, d\}$. Without loss of generality, we may assume that $0 \leq y \leq x \leq 4$.

If $x = 0$, then $y = 0$ and $s = 3154$ which has no 0 among its digits.

If $x = 1$, then $y = 0, 1$ and $s = 3154$ while m has no 4 among its digits.

If $x = 2$, then $s = 3156 + y^y$ and it is easily verified that s has no 2 among its digits for $y = 0, 1, 2$.

If $x = 3$, then $s = 3179 + y^y$ and it is easily verified that s has no 5 among its digits for $y = 0, 1, 2, 3$.

If $x = 4$, then $s = 3408 + y^y$ and, again, it is readily checked that s has no 5 among its digits when $y = 0, 1, 2, 4$. However, when $y = 3$, $s = 3435$ which is clearly a solution.

To summarize, $m = 3435$ is the only solution.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; LUKE LAMOTHE, student, St. Joseph Scollard Hall S.S., North Bay, Ontario, KATHLEEN E. LEWIS, SUNY Oswego. Oswego, NY, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; JOHN GRANT McLOUGHLIN, Okanagan University College, Kelowna British Columbia; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Of the twenty-six solvers (including the proposer), nine of them only gave the answer 3435. *About half of all the solvers claimed, with or without proof, that* 3435 *is the only solution. Chronis, Hess, and Janous found the answer by computer search. Hess remarked that no other solutions were found for the present problem and the corresponding problem on* 5*–digit integers. Janous investigated the corresponding* n*–digit problem of finding all* n*–digit integers* $a_{n-1}a_{n-2}\ldots a_1a_0$ *which equal* $\sum_{k=0}^{n-1} a_k^{a_k}$. *He showed that a necessary condition is* $n \leq 10$. *For* $n > 1$, *he conducted an extensive, but not exhaustive, computer search, which revealed no solutions other than the one found by all the solvers! He made a guess that* 1 *and* 3435 *are the only integers with the desired property. Can any reader prove or disprove this?*