PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 September 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2201. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Let L, M, N be the midpoints of DB, BC, CA respectively. Suppose that AL, OM, DN are concurrent. Show that

$$\text{either} \quad AD \parallel BC \qquad \text{or} \qquad [ABCD] = 2[OBC],$$

where $[\mathcal{F}]$ denotes the area of figure $\mathcal{F}$.

2202. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose than $n \geq 3$. Let $A_1, \ldots, A_n$ be a convex n-gon (as usual with interior angles $A_1, \ldots, A_n$).

Determine the greatest constant C_n such that

$$\sum_{k=1}^{n} \frac{1}{A_k} \geq C_n \sum_{k=1}^{n} \frac{1}{\pi - A_k}.$$

Determine when equality occurs.

2203. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $ABCD$ be a quadrilateral with incircle $\mathcal{I}$. Denote by P, Q, R and S, the points of tangency of sides AB, BC, CD and DA, respectively with $\mathcal{I}$.

Determine all possible values of $\angle(PR, QS)$ such that $ABCD$ is cyclic.

2204. *Proposed by Šefket Arslanagić, Berlin, Germany.*

For triangle ABC such that $R(a+b) = c\sqrt{ab}$, prove that

$$r < \frac{3}{10}a.$$

Here, a, b, c, R, and r are the three sides, the circumradius and the inradius of $\triangle ABC$.

2205. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Find the least positive integer n such that the expression

$$\sin^{n+2} A \ \sin^{n+1} B \ \sin^{n} C$$

has a maximum which is a rational number (where A, B, C are the angles of a variable triangle).

2206. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let a and b denote distinct positive real numbers.

(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}\left(a^p b^{1-p} + a^{1-p} b^p\right) < 4p(1-p)\sqrt{ab} + (1 - 4p(1-p))\,\frac{a+b}{2}.$$

(b) Use (a) to deduce Pólya's Inequality:

$$\frac{a-b}{\log a - \log b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right).$$

Note: "log" is, of course, the natural logarithm.

2207. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Let p be a prime. Find all solutions in positive integers of the equation:

$$\frac{2}{a} + \frac{3}{b} = \frac{5}{p}.$$

2208. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

1. Find a set of positive integers $\{x, y, z, a, b, c, k\}$ such that

$$\begin{aligned} y^2z^2 &= a^2 + k^2 \\ z^2x^2 &= b^2 + k^2 \\ x^2y^2 &= c^2 + k^2 \end{aligned}$$

2. Show how to obtain an infinite number of distinct sets of positive integers satisfying these equations.

2209. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let $ABCD$ be a cyclic quadrilateral having perpendicular diagonals crossing at P. Project P onto the sides of the quadrilateral.

1. Prove that the quadrilateral obtained by joining these four projections is inscribable and circumscribable.

2. Prove that the circle which passes through these four projections also passes through the mid-points of the sides of the given quadrilateral.

2210*. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Given $a_0 = 1$, the sequence $\{a_n\}$ $(n = 1,\ 2,\ \ldots)$ is given recursively by

$$\binom{n}{n}a_n - \binom{n}{n-1}a_{n-1} + \binom{n}{n-2}a_{n-2} - \ldots \pm \binom{n}{\lfloor \frac{n}{2} \rfloor}a_{\lfloor \frac{n}{2} \rfloor} = 0.$$

Which terms have value 0?

2211. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Several people go to a pizza restaurant. Each person who is "hungry" wants to eat either 6 or 7 slices of pizza. Everyone else wants to eat only 2 or 3 slices of pizza each. Each pizza in the restaurant has 12 slices.

It turns out that four pizzas are not sufficient to satisfy everyone, but that with five pizzas, there would be some pizza left over.

How many people went to the restaurant, and how many of these were "hungry"?

2212. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $S = \{1,\ 2,\ \dots,\ n\}$ where $n \geq 3$.

(a) In how many ways can three integers x, y, z (not necessarily distinct) be chosen from S such that $x + y = z$? (Note that $x + y = z$ and $y + x = z$ are considered to be the same solution.)

(b) What is the answer to (a) if x, y, z must be distinct?

2213. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

A generalization of problem 2095 [1995: 344, 1996: 373].

Suppose that the function $f(u)$ has a second derivative in the interval (a, b), and that $f(u) \geq 0$ for all $u \in (a, b)$. Prove that

1. $(y - z)f(x) + (z - x)f(y) + (x - y)f(z) > 0$ for all $x, y, z \in (a, b)$, $z < y < x$

   if and only if $f''(u) > 0$ for all $u \in (a, b)$;

2. $(y - z)f(x) + (z - x)f(y) + (x - y)f(z) = 0$ for all $x, y, z \in (a, b)$, $z < y < x$

   if and only if $f(u)$ is a linear function on (a, b).

Correction

2137. [1996: 124, 317] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{rR}{R + r}$, [**NB**: r instead of 2]

(ii) T lies on the line segment joining the centres of the circumcircle and the incircle of $\triangle ABC$.