THE ACADEMY CORNER

No. 8

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In this issue, courtesy of Waldemar Pompe, student, University of Warsaw, Poland, we print an international contest paper for university students. Please send me your nice solutions.

INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS July 31 – August 1996, Plovdiv, Bulgaria

First day — August 2, 1996

1. Let for $j = 0, 1, \ldots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}.$$

   Calculate $\det A$ — the determinant of A.

2. Evaluate the integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\,\sin x}\,dx\,,$$

   where n is a natural number.

3. A linear operator A on a vector space V is called an *involution* if $A^2 = E$, where E is the identity operator on V.

   Let $\dim V = n < \infty$.

   (i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

   (ii) Find the maximal number of distinct pairwise commuting involutions on V.

4. Let $a_1 = 1$, $a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$.

Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$;

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

5. (i) Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0, 1]$.

Prove that

$$\lim_{n\to\infty} n \int_0^1 (1 + ax + bx^2)^n \, dx = \begin{cases} -1/a & \text{if } a < 0; \\ +\infty & \text{if } a \geq 0. \end{cases}$$

(ii) Let $f\colon [0, 1] \to [0, \infty)$ be a function with continuous second derivative and $f''(x) \leq 0$ for every $x \in [0, 1]$. Suppose that

$$L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, dx$$

exists and $0 < L < +\infty$.

Prove that f' has a constant sign and $L = \left(\min_{x\in[0,1]} |f'(x)| \right)^{-1}$.

6. *Upper content* of a subset E of the plane $\mathbb{R} \times \mathbb{R}$ is defined as

$$\mathcal{C}(E) = \inf \left\{ \sum_{i=1}^{n} \text{diam}\,(E_i) \right\}$$

where the infimum is taken over all finite families of sets $E_1, E_2, \ldots, E_n$ in $\mathbb{R} \times \mathbb{R}$ such that $E \subset \bigcup_{i=1}^{n} E_i$.

Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\text{length}\,(L)\}$$

such that L is a closed line segment onto which E can be contracted.

Show that

(i) $\mathcal{C}(L) = \text{length}\,(L)$, if L is a closed line segment.

(ii) $\mathcal{C}(E) \geq \mathcal{K}(E)$.

(iii) equality in (ii) need not hold even if E is compact.

Hint: If $E = T \cup T'$ where T is the triangle with vertices $(-2, 2)$, $(2, 2)$ and $(0, 4)$, and T' is its reflection about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks:

All distances used in this problem are Euclidean.

Diameter of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x, y) \mid x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f\colon E \to F$ such that $\operatorname{dist}(f(x), f(y)) \leq \operatorname{dist}(x, y)$ for all $x, y \in E$.

A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, that is such that $f(E) = F$.

Triangle is defined as the union of the three line segments joining its vertices, so that it does not contain the interior.

Problems 1 and 2 are worth 10 points, problems 3 and 4 are worth 15 points, problems 5 and 6 are worth 25 points.

You have 5 hours.

Please write the solutions on separate sheets of paper. Good luck!