A Probabilistic Approach to Determinants with Integer Entries

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It is well known that the probability for an integer number to be odd is equal to the probability for the number to be even. What about determinants? What is the probability for a rectangular matrix with integer entries to have odd determinant? More generally, if m is a natural number, what is the probability for which $\det A \equiv m_i \bmod m$, where m_i is chosen from the set $\{0, 1, 2, \ldots, m-1\}$?

I have the following problem to propose; I hope you will find it interesting.

Let A be an $n \times n$ matrix whose elements are integers. What is the probability the determinant of A is an odd number?

Solution:

Let $A = [a_{ij}]$, $i, j = 1, \ldots, n$. It is obvious that

$$\det A \equiv \det\left([a_{ij} \bmod 2]\right) \bmod 2.$$

So the problem is to find the probability that the determinant of an $n \times n$ matrix with elements from the set $\{0, 1\}$ is an odd number. Let A_n be an $n \times n$ matrix with elements from the set $\{0, 1\}$. Let also $N(\det A_n)$ be the number of odd $n \times n$ determinants, and $P(\det A_n)$ be the corresponding probability. Let

$$f(K = \{k_1, k_2, \ldots, k_n\}) = N\left(\begin{vmatrix} k_1 & k_2 & .. & .. & k_n \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ : & : & & & : \\ : & : & & & : \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{vmatrix}\right),$$

where $k_1, k_2, \ldots, k_n \in \{0, 1\}$ are fixed.

Then $N(\det A_n) = \sum(f(K))$, where the sum is calculated for all the $2^n - 1$ possible permutations $K = \{k_1, k_2 \ldots, k_n\}$ with $k_i \in \{0, 1\}$, $i = 1, \ldots, n$ and $k_1, k_2, \ldots, k_n$ not all zero. (Note that $f(0, 0, \ldots, 0) = 0$.)

Lemma: $f(K) = 2^{n-1}N(\det A_{n-1})$, where $k_1 + k_2 + \ldots + k_n \neq 0$.

Proof. It is well known that a determinant remains unchanged if from the elements of one of its columns we substract the corresponding elements of another column. It is also obvious that the same is true for $N(\det A_n)$. Additionally, a determinant just changes sign if we interchange two columns, while $N(\det A_n)$ remains unchanged. So

$$N\left(\begin{vmatrix} k_1 & k_2 & .. & .. & k_n \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{vmatrix}\right) = N\left(\begin{vmatrix} 1 & 0 & .. & .. & 0 \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{vmatrix}\right)$$

$$\iff f(K) = 2^{n-1}N(\det A_{n-1}).$$

Hence $N(\det A_n) = 2^{n-1}N(\det A_{n-1})(2^n - 1)$.
Of course $N(\det A_1) = 1$ and so

$$N(\det A_n) = 2^{n(n-1)/2}\prod_{i=1}^{n}(2^i - 1).$$

Finally, $P(\det A_n) = \frac{N(\det A_n)}{2^{n^2}} \iff P(\det A_n) = \prod_{i=1}^{n}(1 - 2^{-i})$.

Note: The infinite sequence $\prod_{i=1}^{n}(1 - 2^{-i}),\ n = 1, 2, \ldots$ is decreasing and bounded below by 0, so $\lim_{n\to\infty} P(\det A_n)$ exists. Using **Mathematica**, we found that

$$\lim_{n\to\infty} P(\det A_n) \cong 0.2887880950866024212788997219 29\ldots .$$

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