Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

E U R E K A

Vol. 1, No. 7

September 1975

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton

Published by Algonquin College

Send all communications to the editor: Léo Sauvé, Math-Architecture, Algonquin College, Col. By Campus, 281 Echo Drive, Ottawa, Ont., K1S 5G2.

* * *

A SEMANTIC PARADOX

LÉO SAUVÉ, Algonquin College

One often sees the name *paradox* given to statements like "There is an exception to every rule including this one", and to unanswerable questions about a barber who shaves all those and only those who do not shave themselves. But they are not really paradoxes. The given statement refutes itself, and is thus merely false; whereas the barber, in whose description enter contradictory notions, simply cannot exist any more than a square circle can.

Authentic paradoxes are usually divided into two categories: the mathematical and the semantic. The antinomies of set theory, the best known of which is *Russell's Paradox (*1902*)*, are familiar examples of mathematical paradoxes. The paradox of Epimenides (sixth century B.C.), one form of which is the statement "This statement I am now making is a lie", is one example of a semantic paradox. The purpose of this note is to present another semantic paradox which seems to be less well-known. It was formulated by the German mathematician Kurt Grelling in 1908.

If an adjective itself has the property which it describes, it will be said to be *autological*. For example, *short* is autological since it is short, having only one syllable. Other examples are *English*, *français*, *deutsch*, etc. All other adjectives will be called *heterological*, so that the two classes of adjectives are both disjoint and exhaustive. Grelling's paradox consists in the following two contradictory propositions, which are easily derivable from the given definitions:

1) *Heterological* is autological,
2) *Heterological* is heterological.

Interesting discussions on various paradoxes, including Grelling's, can be found in Kleene [1] and L'Abbé [2].

Most adjectives are of course heterological, and autological ones are not easy to find. The author of this note knows a few other than those mentioned above, but he would like to know many more. In Problem 61 below he invites readers to help him in the search for that *rara avis*, the autological adjective.

REFERENCES

1. S.C. Kleene, *Introduction to Metamathematics*, Van Nostrand, 1952.

2. Maurice L'Abbé, *Quelques aspects des mathématiques contemporaines*, Beauchemin, 1963.

* * *

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 55.

For the problems given below, solutions, if available, will appear in EUREKA No. 10, *to be published around December* 15, 1975. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than December* 1, 1975.

Corrections:

1. The deadline for solutions to Problems 51 to 60 is November 1, 1975, not October 1, 1975 as stated in the last issue.

2. Dans le Problème 57, on lira: un groupe d'ordre pn, au lieu de: un groupe d'ordre p^n.

61. *Proposed by Léo Sauvé, Algonquin College.*

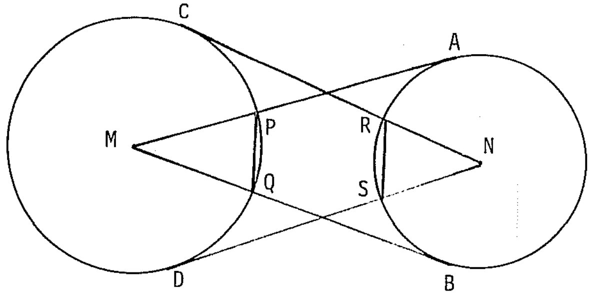
Find autological adjectives other than those given in the article on page 55 of this issue.

62. *Proposed by F.G.B. Maskell, Algonquin College.*

Prove that if two circles touch externally, their common tangent is a mean proportional between their diameters.

63. *Proposed by H.G. Dworschak, Algonquin College.*

From the centres of each of two nonintersecting circles tangents are drawn to the other circle, as shown in the diagram on the following page. Prove that the chords PQ and RS are equal in length. (I have been told that this problem originated with Newton, but have not been able to find the exact reference.)



64. *Proposé par Léo Sauvé, Collège Algonquin.*

Décomposer 10,000,000,099 en un produit d'au moins deux facteurs.

65. *Proposed by Viktors Linis, University of Ottawa.*

Find all natural numbers whose square (in base 10) is represented by odd digits only.

66. *Proposed by John Thomas, University of Ottawa.*

What is the largest non-trivial subgroup of the group of permutations on n elements?

67. *Proposed by Viktors Linis, University of Ottawa.*

Show that in any convex $2n$-gon there is a diagonal which is not parallel to any of its sides.

68. *Proposed by H.G. Dworschak, Algonquin College.*

It takes 5 minutes to cross a certain bridge and 1000 people cross it in a day of 12 hours, all times of day being equally likely. Find the probability that there will be nobody on the bridge at noon.

69. *Proposé par Léo Sauvé, Collège Algonquin.*

Existe-t-il une permutation $n \longmapsto a_n$ de l'ensemble N des entiers naturels telle que la série $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converge?

70. *Proposed by Viktors Linis, University of Ottawa.*

Show that for any 13-gon there exists a straight line containing only one of its sides. Show also that for every $n > 13$ there exists an n-gon for which the above statement does not hold.

* * *

C'est donc par l'étude des mathématiques, et *seulement* par elle, que l'on peut se faire une idée juste et approfondie de ce que c'est qu'une science.

AUGUSTE COMTE

SOLUTIONS

21. *Late solution: Maurice Poirier, École Secondaire Garneau.*

22. *Late solution: Maurice Poirier, École Secondaire Garneau.*

25. [1975, *p.* 42] *Proposed by Viktors Linis, University of Ottawa.*

Find the smallest positive value of $36^k - 5^\ell$ where k and ℓ are positive integers.

Comment by F.G.B. Maskell, Algonquin College.

This problem is a special case of the following proposition concerning two consecutive integers the smaller of which is odd:

If m, n, p are positive integers, the smallest positive value of $(2p)^{2m} - (2p-1)^n$ *is* $4p - 1$.

This can easily be proved by using the number base $4p - 2$. The squares and therefore all positive integral powers of both $2p$ and $2p - 1$ have $2p$ and $2p - 1$ respectively for their final digits in the number scale of $4p - 2$, since

$$(2p)^2 = 4p^2 = (4p-2)p + 2p,$$

$$(2p-1)^2 = 4p^2 - 4p + 1 = (4p-2)(p-1) + (2p-1).$$

Hence the final digit of $(2p)^{2m} - (2p-1)^n$ is $2p - (2p-1) = 1$.

Now

$$(2p)^{2m} - 1 = \{(2p)^m - 1\}\{(2p)^m + 1\} \neq (2p-1)^n,$$

since $(2p)^m - 1$ and $(2p)^m + 1$ are both odd and differ by 2, and therefore have no common factor, nor are they both nth powers of integers, and so $(2p)^{2m} - (2p-1)^n \neq 1$.

The smallest positive value of $(2p)^{2m} - (2p-1)^n$ is now easily seen to be $4p - 1$, which is obtained when $m = 1$ and $n = 2$.

31. *Proposed by Léo Sauvé, Algonquin College.*

A driver cruising on the highway observed that the odometer of his car showed 15951 miles. He noticed that this number is palindromic: it reads the same backward and forward.

"Curious", the driver said to himself. "It will be a long time before that happens again." But exactly two hours later the odometer showed a new palindromic number.

What was the average speed of the car in those two hours?

Solution by F.G.B. Maskell, Algonquin College.

The next palindromic number after 15951 must begin with 16, and therefore must end with 61. The number observed by the driver must therefore have been one of 16061, 16161, 16261, etc. Only the first gives a reasonable increase of 110 over

15951 for a two-hour drive. The average speed, therefore, was 55 miles per hour.

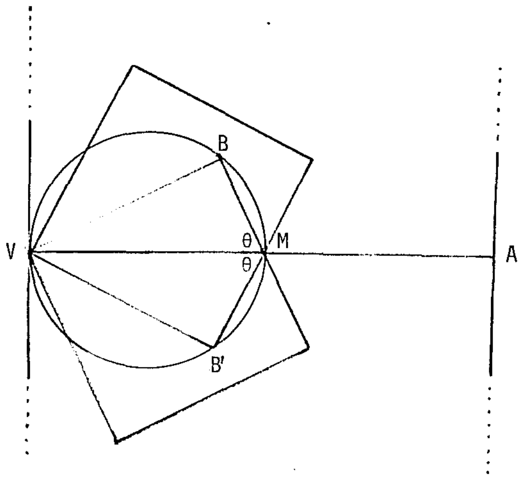
Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; and the proposer.

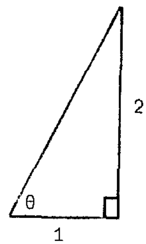
32. *Proposed by Viktors Linis, University of Ottawa.*

Construct a square given a vertex and a midpoint of one side (consider all cases).

Solution by H.G. Dworschak, Algonquin College.

Let V and M be the given vertex and midpoint. If we produce VM to A so that MA = VM, it is clear that the squares of side VA drawn on either side of segment VA are solutions to the problem. To find the other solutions, first construct an angle $\theta = \arctan 2$. This can easily be done as shown in the figure below. Describe





the circle of diameter VM. In the two semicircles draw $\angle VMB = \angle VMB' = \theta$. The squares drawn on VB and VB' with one side going through M are the remaining solutions to the problem. The proofs of these statements are obvious.

Also solved by Michael P. Closs, University of Ottawa; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and Léo Sauvé, Algonquin College.

Editor's comment.

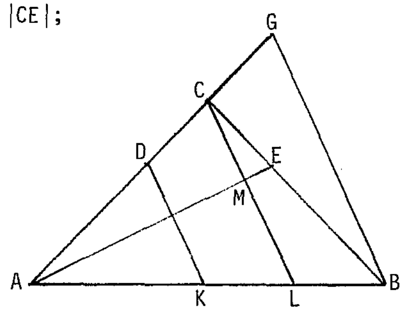
Closs rightly points out that essentially the same method can be used to construct a square given a vertex and any "rational" point on one side (that is, a point whose distance from a vertex is rational relative to the unit measure of one side of the square).

But why stop at "rational" points?

33. *Proposed by Viktors Linis, University of Ottawa.*

On the sides CA and CB of an isosceles right-angled triangle ABC, points D and E are chosen such that $|CD| = |CE|$. The perpendiculars from D and C on AE intersect the hypotenuse AB in K and L respectively. Prove that $|KL| = |LB|$.

Solution by Léo Sauvé, Algonquin College.



Produce DC to G so that $|CG| = |CD| = |CE|$; then $\Delta ACE \equiv \Delta GCB$ and $\angle CAE = \angle CBG$. But if CL meets AE in M, then $\Delta ACE \,|||\, \Delta CME$ and so $\angle MCE = \angle CAE = \angle CBG$, from which $GB \| CL \| DK$. Since C bisects DG, it follows that L bisects KB.

Also solved by G.D. Kaye, Department of National Defence (trigonometric solution); and F.G.B. Maskell, Algonquin College (analytic solution).

34. *Proposed by H.G. Dworschak, Algonquin College.*

Once a bright young lady called Lillian
Summed the numbers from one to a billion.
    But it gave her the fidgets
    To add up the *digits*;
If you can help her, she'll thank you a million.

Solution by Léo Sauvé, Algonquin College.

If we write all the numbers from 0 to $10^n - 1$ in a row twice, once in the usual and once in the reverse order, thus

$$\begin{array}{cccccc} 0 & 1 & 2 & \cdots & 10^n-2 & 10^n-1 \\ 10^n-1 & 10^n-2 & 10^n-3 & \cdots & 1 & 0 \end{array}$$

we observe that the sum of the digits is the same in every column, and that sum is $9n$ for $10^n - 1$ contains n 9's. Since there are 10^n columns, the sum of the digits of all the numbers from 0 to $10^n - 1$ is $\frac{1}{2} \cdot 10^n \cdot 9n = 45n \cdot 10^{n-1}$, and the required sum of the digits of all the numbers from 1 to 10^n is $45n \cdot 10^{n-1} + 1$. For Lillian's billion, we have $n = 9$, and the sum of the digits is 40,500,000,001.

Also solved by André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment.

Some solvers interpreted the problem differently, and found instead the sum of the digits of the sum of the numbers, which is not much of a problem. This was to be expected, for it is a hallmark of all great poetry to be susceptible of various interpretations.

Maskell pointed out that the answer would be different if Lillian were

British, but it is clear this possibility must be rejected. For to sum the numbers from 1 to a British billion (one million million) at the rate of one number per second, twenty-four hours a day, without ever stopping even for a cup of tea, would take approximately 31,700 years.

35. *Proposed by John Thomas, Digital Methods Ltd.*

Let m denote a positive integer and p a prime. Show that if $p \mid m^4 - m^2 + 1$, then $p \equiv 1 \pmod{12}$.

Solution by Viktors Linis, University of Ottawa.

Let $N = m^4 - m^2 + 1$. If p divides N, then

$$4N = 4m^4 - 4m^2 + 4 = (2m^2 - 1)^2 + 3 \equiv 0 \pmod{p},$$

which implies that -3 is a quadratic residue. It follows that $p \equiv 1 \pmod 6$ (see Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 96, p. 75). Also

$$N = (m^2 + 1)^2 - 3m^2 \equiv 0 \pmod{p}$$

implies that $3m^2$ is a quadratic residue for *all* m; hence 3 is also a quadratic residue mod p.

Note that $N \equiv 1 \pmod{12}$ because $N - 1 = m^4 - m^2 = (m-1)\,m\,m\,(m+1)$ has two even factors and at least one divisible by 3. The divisors of $N \pmod{12}$ can be either $\equiv 1$ or $\equiv 7$ but if a prime $p \equiv 7$ then 3 is a non-residue, which leaves only the case $p \equiv 1 \pmod{12}$.

36. *Proposé par Léo Sauvé, Collège Algonquin.*

Si m et n sont des entiers positifs, montrer que

$$\sin^{2m}\theta \cos^{2n}\theta \le \frac{m^m n^n}{(m+n)^{m+n}},$$

et déterminer les valeurs de θ pour lesquelles il y a égalité.

I. *Solution by Viktors Linis, University of Ottawa.*

Let $\sin^2\theta = x$; then $f(x) = x^m (1-x)^n$ has a unique maximum in $[0,1]$ at $x_0 = \dfrac{m}{m+n}$. Hence $f(x) \le f(x_0) = \dfrac{m^m n^n}{(m+n)^{m+n}}$ which holds for all positive real numbers m, n (not just integers). The equality is attained for $\sin^2\theta = \dfrac{m}{m+n}$ or equivalently for $\theta = \pm \operatorname{Arctan}\sqrt{m/n} + k\pi$ where Arctan denotes the principal value and k is any integer.

II. *Solution by the proposer.*

We will prove the following related inequality, valid for $x > 0$, $y > 0$, which is interesting in its own right and which can be proved by purely algebraic means when m and n are positive integers:

$$\frac{x^m y^n}{(x+y)^{m+n}} \le \frac{m^m n^n}{(m+n)^{m+n}}, \tag{1}$$

equality occurring only when $\frac{x}{m}=\frac{y}{n}$. The desired inequality will then result if we set $x=\sin^2\theta$, $y=\cos^2\theta$ in (1), equality occurring only when $\frac{\sin^2\theta}{m}=\frac{\cos^2\theta}{n}$, that is, only when $|\tan\theta| = \sqrt{m/n}$.

If we apply the theorem of the means G.M. ≤ A.M. to m numbers $\frac{x}{m}$ together with n numbers $\frac{y}{n}$, we get

$$\sqrt[m+n]{\left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n} \le \frac{x+y}{m+n},$$

equality occuring only when $\frac{x}{m}=\frac{y}{n}$, whence

$$\frac{x^m y^n}{m^m n^n} \le \frac{(x+y)^{m+n}}{(m+n)^{m+n}},$$

and (1) follows.

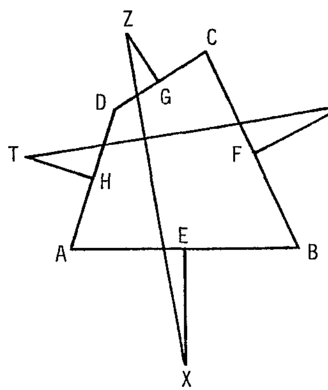
Also solved by G.D. Kaye, Department of National Defence; and F.G.B. Maskell, Algonquin College.

37. *Proposed by Maurice Poirier, École Secondaire Garneau.*

E, F, G, and H are the midpoints of sides AB, BC, CD, and DA, respectively, of the convex quadrilateral ABCD. EX, FY, GZ, and HT are drawn externally perpendicular to AB, BC, CD, and DA, respectively, and EX = ½AB, FY = ½BC, GZ = ½CD, and HT = ½DA. Prove that XZ = YT and XZ ⊥ YT.

Solutions by Léo Sauvé, Algonquin College.

I. The quadrilateral need not be convex and some or all of the points A, B, C, D can be collinear, provided the vertices of Δ's AXB, BYC, CZD, and DTA are in the same, say counterclockwise, order. As will be clear from the solution below, the theorem holds even if two or more of the points A, B, C, D coincide.



Imagine the figure to be embedded in the complex plane, and let the points A, B, C, D, X, Y, Z, T correspond to the complex numbers a, b, c, d, x, y, z, t, respectively. Since a counterclockwise rotation of $\frac{\pi}{2}$ takes $\overrightarrow{XB}$ into $\overrightarrow{XA}$, we have $a-x=i(b-x)$, so that

$$x = \frac{a - ib}{1 - i},$$

and similarly

$$y = \frac{b - ic}{1 - i}, \quad z = \frac{c - id}{1 - i}, \quad t = \frac{d - ia}{1 - i}.$$

Since $t - y = i(z - x)$, we conclude that $XZ = YT$ and $XZ \perp YT$.

II. We represent the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, $\overrightarrow{EX}$, $\overrightarrow{FY}$, $\overrightarrow{GZ}$, and $\overrightarrow{HT}$ by $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\overrightarrow{a'}$, $\overrightarrow{b'}$, $\overrightarrow{c'}$, and $\overrightarrow{d'}$, respectively, so that (make a figure!)

$$\overrightarrow{XZ} = -\overrightarrow{a'} + \tfrac{1}{2}\vec{a} + \vec{b} + \tfrac{1}{2}\vec{c} + \overrightarrow{c'}$$

and

$$\overrightarrow{YT} = -\overrightarrow{b'} + \tfrac{1}{2}\vec{b} + \vec{c} + \tfrac{1}{2}\vec{d} + \overrightarrow{d'}.$$

Since $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$, we have

$$\tfrac{1}{2}\vec{a} + \vec{b} + \tfrac{1}{2}\vec{c} = \tfrac{1}{2}\vec{b} - \tfrac{1}{2}\vec{d},$$
$$\tfrac{1}{2}\vec{b} + \vec{c} + \tfrac{1}{2}\vec{d} = \tfrac{1}{2}\vec{c} - \tfrac{1}{2}\vec{a}$$

and we can write

$$\overrightarrow{XZ} = -\overrightarrow{a'} + \overrightarrow{c'} + \tfrac{1}{2}\vec{b} - \tfrac{1}{2}\vec{d},$$
$$\overrightarrow{YT} = -\overrightarrow{b'} + \overrightarrow{d'} + \tfrac{1}{2}\vec{c} - \tfrac{1}{2}\vec{a}.$$

Now a counterclockwise rotation of $\frac{\pi}{2}$ takes $\overrightarrow{XZ}$ into

$$(\overrightarrow{XZ})^{\perp} = -\tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{c} - \overrightarrow{b'} + \overrightarrow{d'} = \overrightarrow{YT},$$

and it follows that $XZ = YT$ and $XZ \perp YT$.

Also solved by G.D. Kaye, Department of National Defence (trigonometric solution); and André Ladouceur, École Secondaire De La Salle (analytic solution).

Editor's comment.

This theorem is due to H. Van Aubel, who was a professor in the *athénée* of Antwerp around 1880. It appears as Problem 10 on page 23 of Coxeter [1] and a solution (different from those above) is given at the back of the book. Kelly [2] generalized it to four non-coplanar points.

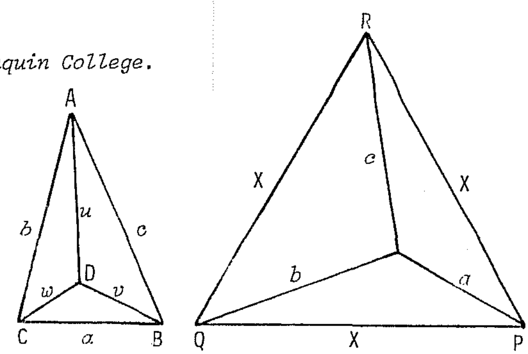
REFERENCES

1. H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1969.
2. Paul J. Kelly, *Von Aubel's Quadrilateral Theorem*, Mathematics Magazine, Vol. 39 (1966), p. 35.

38. *Proposed by Léo Sauvé, Algonquin College.*

Consider the two triangles ΔABC and ΔPQR shown at the right. In ΔABC, $\angle ADB = \angle BDC = \angle CDA = 120^\circ$. Prove that $X = u + v + w$.

(Third USA Mathematical Olympiad - May 7, 1974)



Solution by G.D. Kaye, Department of National Defence.

In the plane of ΔABC construct three parallelograms with common vertex at D, as shown in Figure 1. Each parallelogram has angles of 60° and 120°. Rearrange the parallelograms so that the vertices of their acute angles meet in a point, with 60° between each pair of adjacent parallelograms, as shown in Figure 2. If the outer vertices of the parallelograms are renamed P', Q', and R', it is clear that the outer sides of the parallelograms coincide with the sides of $\Delta P'Q'R'$. This $\Delta P'Q'R'$ is clearly identical to the given ΔPQR and the length of its side is $X = u + v + w$.

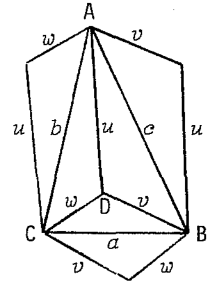


Figure 1

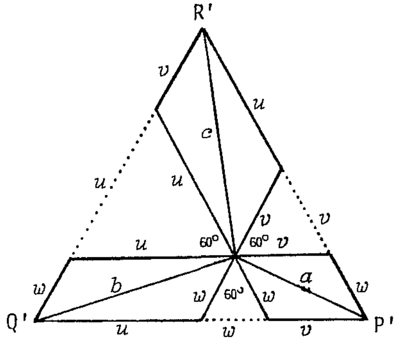


Figure 2

Also solved by Léo Sauvé, Algonquin College.

Editor's comment.

Three different solutions of this problem were given in *The Mathematics Teacher*, Vol. 68, No. 1 (Jan. 1975), in a report on the Third USA Mathematical Olympiad written by Samuel L. Greitzer, Chairman of the Olympiad Committee. Greitzer states: "This problem seems to have aroused a good deal of interest. Several people who had read the problems in the *New York Times* article sent in solutions, using methods ranging from analytic geometry through trigonometry and vectors."

In this editor's opinion, the solution given here is superior to the three published ones.

39. *Proposé par Maurice Poirier, École Secondaire Garneau.*

On donne un point P à l'intérieur d'un triangle équilatéral ABC tel que les longueurs des segments PA, PB, PC sont 3, 4, et 5 unités respectivement. Calculer l'aire du ΔABC.

I. *Solution du proposeur.*

Faisons tourner de 60° le ΔABP autour du point B, de sorte que BA vienne coïncider avec BC, le point P occupant la nouvelle position P', comme l'indique la figure 1. Le $\Delta BPP'$ est équilatéral, de sorte que $PP' = 4$, le $\Delta PP'C$ est rectangle en P', et $\angle BP'C = 150^\circ$. La loi des cosinus appliquée au $\Delta BP'C$ donne alors $\overline{BC}^2 = 25 + 12\sqrt{3}$, et

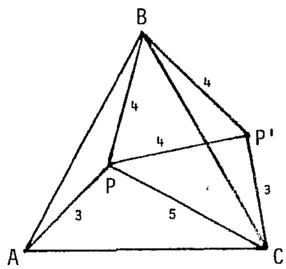


Figure 1

l'aire du Δ ABC est

$$\frac{\sqrt{3}}{4}\overline{BC}^2 = \frac{25\sqrt{3}+36}{4} \doteq 19.825$$

II. *Solution by H.G. Dworschak, Algonquin College.*

More generally, suppose we have given an equilateral Δ ABC, the length of whose side is X, and an interior point P such that PA $= a$, PB $= b$, and PC $= c$, as shown in Figure 2. The area of Δ ABC is then $\frac{\sqrt{3}}{4}X^2$, and we will evaluate this in terms of a, b, and c.

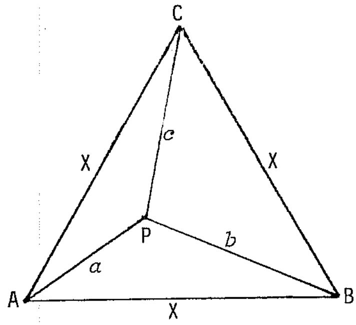


Figure 2

Construct a Δ FGH with sides a, b, c, (such a triangle always exists since, for example, $a+b > X > c$) and let D be the point at which the three sides subtend equal angles of 120°, as shown in Figure 3. If we set DF $= u$, DG $= v$, and DH $= w$, then we know from Problem 38 that $X = u+v+w$.

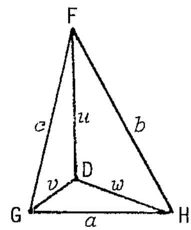


Figure 3

If we apply the law of cosines to the three small triangles in Figure 3, we get

$$a^2 = v^2 + w^2 + vw\ , \qquad (1)$$
$$b^2 = w^2 + u^2 + wu\ , \qquad (2)$$
$$c^2 = u^2 + v^2 + uv\ ; \qquad (3)$$

and if we sum the areas of these same three triangles, the area K of Δ FGH becomes

$$K = \tfrac{1}{2}vw\sin 120^\circ + \tfrac{1}{2}wu\sin 120^\circ + \tfrac{1}{2}uv\sin 120^\circ$$
$$= \frac{\sqrt{3}}{4}(vw+wu+uv)\ ,$$

so that $4\sqrt{3}\,K = 3(vw+wu+uv)$. Adding this last equation to (1), (2), and (3) now gives

$$a^2+b^2+c^2+4\sqrt{3}\,K = 2(u+v+w)^2 = 2X^2\ ,$$

and the required area of Δ ABC is

$$\frac{\sqrt{3}}{4}X^2 = \frac{\sqrt{3}}{8}(a^2+b^2+c^2) + \frac{3}{2}K\ ,$$

where $K = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = \frac{1}{2}(a+b+c)$.

Also solved by Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College; and Léo Sauvé, Algonquin College.

Editor's comment.

Linis and Maskell both gave analytic solutions of the special case treated in solution I, and each pointed out that the area of Δ ABC would be $\frac{1}{4}(25\sqrt{3}-36) \doteq 1.825$ if the point P were *outside* the triangle.

For the more general case treated in solution II, the following question can be asked: Given three positive numbers a, b, and c, does there always exist an equilateral ΔABC and a point P such that $PA = a$, $PB = b$, and $PC = c$? Linis answered this question and gave the condition of existence as

$$a^4 + b^4 + c^4 \le 2(b^2c^2 + c^2a^2 + a^2b^2).$$

40. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $\{a_n\}$ une suite de nombres complexes non nuls pour laquelle il existe un $r > 0$ tel que

$$m \ne n \Rightarrow |a_m - a_n| \ge r.$$

Si $u_n = \dfrac{1}{|a_n|^\alpha}$, où $\alpha > 2$, montrer que la série $\sum_{n=1}^{\infty} u_n$ converge. Que peut-on dire si $\alpha = 2$?

Complete solutions were submitted by Viktors Linis, University of Ottawa; and Léo Sauvé, Algonquin College.

Editor's comment.

The *convergence exponent* of the sequence $r_1, r_2, r_3, \ldots, r_n, \ldots$, where $0 < r_1 \le r_2 \le \cdots$ and $\lim_{n\to\infty} r_n = \infty$, is defined as the number λ having the following property: the series $\sum_{n=1}^{\infty} r_n^{-\sigma}$ converges for $\sigma > \lambda$ and diverges for $\sigma < \lambda$. It is known that

$$\lambda = \limsup_{n\to\infty} \frac{\log n}{\log r_n}.$$

(For a proof of this, see *Problems and Theorems in Analysis*, Vol. 1, by Pólya and Szegö, Springer-Verlag, 1972, p. 202.)

Since an infinite series of positive numbers if convergent is absolutely convergent, we can, rearranging if necessary, assume that the numbers $|a_n|$ are in increasing magnitude,

$$0 < |a_1| \le |a_2| \le |a_3| \le \cdots,$$

and the proposed problem is equivalent to the following:

Prove that the convergence exponent of the sequence $|a_1|, |a_2|, |a_3|, \ldots$ *is at most* 2.

I reproduce below the beautiful proof of this theorem by Pólya and Szegö (op. cit. p. 306).

We enclose each a_ν, $\nu = 1, 2, \ldots, n$, in a circle with centre a_ν and radius $\frac{r}{2}$. These circles have no common inner points and are completely contained in the circle $|z| \le |a_n| + \frac{r}{2}$. Therefore

$$n\pi\frac{r^2}{4} < \pi(|a_n| + \frac{r}{2})^2,$$

and it follows immediately that

$$\lambda = \limsup_{n\to\infty} \frac{\log n}{\log|a_n|} \le 2.$$

The fact is that the convergence exponent is *exactly* 2, since the given series $\sum_{n=1}^{\infty} u_n$ may converge or diverge if $\alpha = 2$. The proof of this statement which follows is taken from the solution submitted by Linis.

If all terms of the sequence $\{a_n\}$ are real and $a_n = nr$, then

$$\sum_{n=1}^{\infty} u_n = \frac{1}{r^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

For the divergence case let us arrange the a_n's as follows: each circle $|z| = jr$, $j = 1, 2, \ldots$, has $6j$ points which lie on the rays from the origin with argument $\frac{2\pi k}{6j}$ $(k = 0, 1, \ldots, 6j - 1)$. The distance between points on different circles is certainly $\ge r$, and on the same circle the distances are $d_j = 2jr\sin\frac{\pi}{6j}$. Now d_j as a function of j is increasing (check the derivative!); hence $d_j \ge d_1 = 2r\sin\frac{\pi}{6} = r$. Since the jth circle contains $6j$ points of the sequence, we have

$$\sum_{n=1}^{\infty} u_n = \sum_{j=1}^{\infty} \frac{6j}{(jr)^2} = \frac{6}{r^2} \sum_{j=1}^{\infty} \frac{1}{j},$$

which diverges.

* * *

ARS MAGNA

Mathematicians have always been fascinated by word games. I do not speak of puns, which are commonly called the lowest form of wit, and even that is too much praise by half. Two word games which are closer to a mathematician's heart are the palindromes and the anagrams.

A palindrome is a finite sequence of letters with central symmetry; thus a sequence of letters $i \mapsto a_i$, $i = 0, 1, 2, \ldots, n$, is a palindrome if $a_i = a_{n-i}$ for all i. Some palindromes have become trite; for example, Napoleon's last lament:

Able was I ere I saw Elba;

or the first words spoken in the Garden of Eden:

Madam, I'm Adam.

Others, though difficult technical feats, merely amuse or startle us; for example,

Ban campus motto: "Bottoms up, MacNab!"

Step on no pets!

Eve saw diamond, erred, no maid was Eve.

Stop! Murder us not tonsured rumpots!

Doc, note, I dissent. A fast never prevents a fatness. I diet on cod.

Straw? No, too stupid a fad. I put soot on warts.

Finally, there are a few sublime examples which tug at the heart and stir the emotions:

Niagara, O roar again!

A man, a plan, a canal - Panama.

Are we not drawn onwards, we Jews, drawn onward to new era?

But the rigidity imposed by the requirement of central symmetry is such that the palindrome can never be anything but a minor, though difficult, linguistic art.

The anagram, on the other hand, which was invented by the Greek poet Λυκοφρων in 260 B.C., has a built-in capacity for greatness. Since it is simply a permutation of a finite set of letters, a given phrase of n letters gives rise to $n!$ different anagrams, in most of which, of course, the permuted phrase is meaningless. It takes skill to select a permuted phrase which is meaningful; but one must be an artist to select a meaningful permuted phrase *which is closely related to the original phrase*. Here are some outstanding examples:

Desperation.
A rope ends it.

The burning of Ancient Rome.
Fire, mob, Nero chanting tune.

Abandon hope, all ye who enter here.
Hear Dante! Oh, beware yon open hell.

Finally, the anagram can achieve the status of an *ars magna* which, fittingly enough, is a permutation of the word *anagrams*, when the two related phrases have the additional constraint of a preimposed subject. Here are a few gems on mathematics:

Eleven + two.
Twelve + one.

A number line.
Innumerable.

A decimal point.
I'm a dot in place.

Integral calculus.
Calculating rules.

Higher mathematics.
M.A. teaches him right.

Augustus De Morgan.
Great gun! Do us a sum!

I invite the readers of EUREKA to achieve greatness by sending me *original* anagrams on the subject of mathematics. I will from time to time publish the best ones (they will make ideal "fillers") with credit given to the sender.

All of the palindromes and anagrams cited in this article were taken from *Palindromes and Anagrams*, by Howard W. Bergerson (Dover, 1973), which contains literally thousands of them.

The Editor.