The Relocation Property

Charles Ledger and Kim Langen, Spirit of Math Schools

Introduction

When working in the intermediate level classroom in the North York School Board it became critically clear that students lacked basic foundational numeracy skills. They did not understand how to work with numbers in an efficient manner, or even understand how numbers could be manipulated. When the Relocation Property was introduced students were able to very quickly master numeracy and consequently we were able to do significantly more with the students. This property has now been introduced to students as young as grade 3, with components introduced to grade 1 students. In this vignette the Relocation Property will be quickly described, and the work with students from grades 1 through 7 will be shared.

The Relocation Property

In English, we read from left to right. We usually do mathematics in the same direction, but sometimes the calculations involved make this difficult. The relocation property helps us to rearrange such calculations so that they are easier to do from left to right.

The relocation property is an important property that students will use when they learn algebra. It is introduced long before the students begin algebra, so that when they learn algebra, relocation will be second nature to them.

Since the aim of this property is to improve numeracy skills, calculators must not be used.

The Relocation Property: Addition & Subtraction

To calculate the answer to 16 + 7 + 2 + 19 + 8 + 33 + 1 + 14, we can add the numbers in any order. However, rearranging addition or multiplication questions allows us to calculate faster. In this addition example, we might choose pairs that add up to multiples of 10. Thus, we would choose 16 + 14, then 7 + 33, then 2 + 8, and 19 + 1. Adding the results of these smaller questions gives the final answer of 100:

$$\begin{aligned}16 + 7 + 2 + 19 + 8 + 33 + 1 + 14 &= 16 + 14 + 7 + 33 + 2 + 8 + 19 + 1\\ &= 30 + 40 + 10 + 20\\ &= 100\end{aligned}$$

Adding the numbers in a different order allows us to calculate more easily what seems like a difficult question. The problem comes when we need to subtract. Consider the expression 19 + 26-3—9 + 4 + 43. We cannot rearrange the numbers for easier calculation, because we have addition and subtraction. Here is where the relocation property comes to the rescue. We will consider each operation sign to be glued to the number that follows it. The Relocation Property states:

The operation sign goes with the number that follows it.

Now it is easy to rearrange the numbers in our example. We have a 19, a plus 26, a minus 3, a minus 9, a plus 4, and a plus 43. Rearranging to end up with as many multiples of ten as possible gives us 19 - 9, added to 26 + 4, added to 43 - 3:

$$\begin{aligned}19 + 26 - 3 - 9 + 4 + 43 &= 19 - 9 + 26 + 4 + 43 - 3\\ &= 10 + 30 + 40\\ &= 80\end{aligned}$$

The Relocation Property: Multiplication & Division

The relocation property also works for multiplication and division. We simply remember that a multiplication or division sign must stay with the number following it when we do the rearranging. For example, for the expression

$$33 \times 13 \div 2 \div 11 \div 13 \times 10$$

One possible rearrangement that makes calculations easier is 33÷11, multiplied by 13÷13, multiplied by 10÷2.

$$\begin{aligned} 33 \times 13 \div 2 \div 11 \div 13 \times 10 &= 33 \div 11 \times 13 \div 13 \times 10 \div 2 \\ &= 3 \times 1 \times 15 \\ &= 15 \end{aligned}$$

You may have noticed that, in these examples, we have kept addition and subtraction separate from multiplication and division. There is a good reason for this. When a question involves all four operations, then all multiplication and division must be performed before any addition or subtraction. This is called order of operations. For the purposes of this discussion however, we just want to practice rearranging numbers with their signs without worrying about order of operations. The combination of the four operations can be explained in another session under order of operations.

Identity Elements

Consider this expression: 5÷11 x 22. There are two signs, and they each go with the numbers that follow them. So we have ÷11 and x 22. But what sign goes with the 5? Since the question involves only multiplication and division, the sign must be a multiplication or division sign. But which is it? Well, we can write something at the beginning of any multiplication or division expression that will include an extra sign, but will not change its value. We can write 1 x in front of the expression:

$$5 \div 11 \times 22 = 1 \times 5 \div 11 \times 22$$

We see that the sign in front of the 5 is a multiplication sign.

The number 1 is called the identity element for multiplication, because putting 1 x in front of an expression does not change its value. To rearrange the sample expression above, we simply start with 1, and then do the rest of the operations in any order we want. However, if you think about it, we cannot start by dividing, so we must choose either x 5 or x 22 as our first operation. Although either one will work, x 22 is a better choice than x 5.

Here is why: We know that we will have to divide by 11 at some point, and the sooner the better. This way, we keep the running result as small as possible. Doing divisions as early as possible keeps the running result as small as possible. So the order 1 x 22 ÷ 11 x 5 puts the pair x 22÷11 as close to the front as possible. The only other choices are 1 x 22 x 5÷11 and 1 x 5 x 22÷11, and you will notice that the numbers become larger in each of these orders if you calculate them from left to right. Therefore, the best arrangement for calculation is:

$$\begin{aligned} 5 \div 11 \times 22 &= 22 \div 11 \times 5 \\ &= 2 \times 5 \\ &= 10 \end{aligned}$$

Like multiplication, there is an identity element for addition, the number 0. Zero added to an expression does not change its value. We use this fact when we add and subtract, as in a question like 5 – 8 + 4. There is no sign in front of the 5, but since the question involves only addition and subtraction, we know it must be either a plus or minus sign. The identity element tells us it is a plus sign, for we can write 0 + 5 – 8+ 4 without changing the value of the expression.

Additional Notes:

Teaching Strategy: We found that some students had trouble remembering that operation signs go with the numbers that directly follow them. Writing the numbers in a column helped these students. This forced them to include each operation symbol with its following number. For example, students would write

2÷11 x 4÷3 x 55 x 9 as:

X 2

÷11

x 4

÷3

x 55

x9

Then write 1 x 55÷11 x 2 x 9÷3 x 4. It also helped them when they crossed off each operation/number pair as it was used. Then the students were then able to calculate from left to right to arrive at the answer, 120.

This technique of writing the operations in a column, while effective for students, who are having difficulty, frustrated those who could see the best arrangement by just looking at the question. Normally students were encouraged to skip this column step.

Avoid Fractions: Some teachers and parents proceed immediately to fractions at this stage, but this is strongly discouraged. The purpose is to develop an understanding of the relocation property and give practice with mental arithmetic. Both of these extremely important ideas will be lost (or at least seriously downplayed) if expressions with division are written in fraction form at this point. We have found that students are able to do the relocating that is taught here, as well as the fraction-cancelling that is taught another time.

For those who do not understand what we mean by "written in fraction form", here it is. But don't do it!

Consider the expression:

2÷11 x 4÷3 x 55 x 9.

Written in fraction form it would be:

$$\frac{2 \times 4 \times 55 \times 9}{11 \times 3}$$

Then it is just a matter of cancelling. This is not encouraged at this point because it doesn't help develop the understanding or appreciation of the flexibility of numbers.

Prime Factoring: The ideas behind prime factoring are so useful and pervasive that they sneak into the student exercises in the multiplication questions. Students are introduced to prime numbers and are instructed, for example, to look for prime factors of 2 and 5, and then combine them to make products of 10.

The Peeling Method: Some students find it difficult to assemble the relocations with just the numbers given in the questions and calculate the running totals in their heads. These students cannot keep track of everything mentally. We found it easier to have these students look for prime factors in the divisors as usual, but then split them off from their dividends before relocation. An example will help explain this idea.

36 x 2 ÷ 15 x 50

In this question, as any relocation question, consider the divisions first. There is only one division: a division by 15. Since 15 = 3 x 5, we need to divide by 3 and divide by 5. Where do we find the 3 and the 5? There is a factor of 3 in the 36 and a factor of 5 in the 50. The idea is to peel a 3 off the 36 and a 5 off the 50, like this:

3 x 12 x 2÷15 x 5 x 10

Then we can relocate for easier left-to-right calculation by grouping the 3 and the 5 together to make a multiplication by 15, which we then immediately divide by 15, like this:

3 x 5÷15 x 2 x 12 x 10 = 240

In other words, we find the prime factors we need and split them off from the dividends before relocating them (instead of doing it all mentally, like this: 36 x 50÷15 x 2 = 240). We build a copy of the number that we must divide by out of prime factors 'peeled' off the available dividends. Notice, please, that we do not prime factor every number in the expression. We only peel off the prime factors needed to build the required dividend. Unlike prime factoring with every number, this 'peeling off' method still promotes the development of the mathematical feel for numbers and their factors while making the process of relocation and calculation manageable for students who can't keep track of all the members mentally. Peeling the right factors off still promotes numeracy because the students have to search for factors and peel off only what they need. On the other hand, prime factoring of all numbers completely wrecks this higher level of thinking. Some of your students will realize this and just want to prime factor everything first. Tell them that the challenge is to do each question with the least amount of peeling possible. **Prime factoring everything first is cheating.**

Key Terms:

Expression: a number, or a series of operations with numbers. For example, both 9 and 8÷3 x 1 x 9 are expressions. Basically, any group of calculations that does not contain an equal sign is an expression.

Identity Element: a number related to an operation which, when included in an expression using that operation, does not change the value of the expression. The identity element for multiplication is 1, because multiplying by 1 does not change the value of an expression. The identity element for addition is 0, because adding 0 does not change the value of an expression.

If you're a mathematician, you'll know that 0 is also a right identity for subtraction, and 1 is also a right identity for division. Here, though, we are using 0 and 1 only as identity elements for the operations addition and multiplication, respectively, so that we can show the addition sign in front of expressions containing additions and subtractions or the multiplication sign in front of expressions containing multiplications and divisions. In other words, although 0 is not a full identity element for both addition and subtraction, it can be used to show the addition sign at the front of expressions containing both additions and subtractions. Similarly, 1 can be used to show the multiplication sign in front of expressions containing both multiplications and divisions even though it is not a full identity for both multiplication and division.

Relocation Property: a property of addition and subtraction expressions, and multiplication and division expressions, that states we can rearrange numbers, with the signs preceding them, because the order of calculation does not change the value of an expression. 1 x 2÷14 x 7 = 1 x 7 x 2÷14.

Practice Questions

1. $18 \times 6 \times 21 \div 9 \div 7 \times 2 \div 4 \div 3$
2. $23 - 8 - 12 + 48 + 7 + 2 + 34 - 4$
3. $45 \times 27 \div 9 \times 12 \div 5 \div 3 \div 4 \div 3$
4. $55 \times 13 \div 7 \div 11 \times 6 \div 13 \times 15 \div 5 \times 21 \div 3 \div 9 \div 5$
5. $26 + 73 + 14 - 51 + 151 + 27 - 30$
6. $26 \div 7 \div 12 \times 3 \times 14 \times 2 \div 13 \times 5$
7. $1 \div 2 \div 3 \times 15 \div 5 \div 7 \times 77 \div 11 \times 26 \div 13$

Solutions

1. $18 \times 6 \times 21 \div 9 \div 7 \times 2 \div 4 \div 3$
   $18 \div 9 \times 2 \div 4 \times 21 \div 7 \div 3 \times 6 = 6$

2. $23 - 8 - 12 + 48 + 7 + 2 + 34 - 4$
   $23 + 7 + 48 - 8 + 34 - 4 + 2 - 12 = 30 + 40 + 20$
   $= 90$

3. $45 \times 27 \div 9 \times 12 \div 5 \div 3 \div 4 \div 3$
   $45 \div 9 \div 5 \times 12 \div 3 \div 4 \times 27 \div 3 = 9$

4. $55 \times 13 \div 7 \div 11 \times 6 \div 13 \times 15 \div 5 \times 21 \div 3 \div 9 \div 5$
   $55 \div 11 \div 5 \times 13 \div 13 \times 21 \div 7 \div 3 \times 15 \div 5 \times 6 \div 9 = 2$

5. $26 + 73 + 14 - 51 + 151 + 27 - 30$
   $26 + 14 - 30 + 73 + 27 + 151 - 51 = 10 + 100 + 100$
   $= 210$

6. $26 \div 7 \div 12 \times 3 \times 14 \times 2 \div 13 \times 5$
   $26 \div 13 \times 2 \times 3 \div 12 \times 14 \div 7 \times 5 = 10$

7. $1 \div 2 \div 3 \times 15 \div 5 \div 7 \times 77 \div 11 \times 26 \div 13$
   $1 \times 15 \div 3 \div 5 \times 77 \div 7 \div 11 \times 26 \div 13 \div 2 = 1$

Summary:

We can use the relocation property to rearrange calculations by carrying any operation signs with the numbers that follow them. We can rearrange any addition and subtraction question, or any multiplication and division question. The identity elements for addition and multiplication come in useful in telling us which sign belongs to the first number in an expression, even though no sign is written.