2018 CMO Qualifying Repêchage

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Official Problem Set

1. Determine all real solutions to the following system of equations

$$\begin{cases} y = 4x^3 + 12x^2 + 12x + 3 \\ x = 4y^3 + 12y^2 + 12y + 3 \end{cases}$$

2. We call a pair of polygons, p and q, *nesting* if we can draw one inside the other, possibly after rotation and/or reflection; otherwise we call them *non-nesting*.

   Let p and q be polygons. Prove that we can find a polygon r, which is similar to q, such that r and p are non-nesting if and only if p and q are not similar.

3. Let ABC be a triangle with $AB = BC$. Prove that $\triangle ABC$ is an obtuse triangle if and only if the equation

$$Ax^2 + Bx + C = 0$$

   has two distinct real roots, where A, B, C are the angles in radians.

4. Construct a convex polygon such that each of its sides has the same length as one of its diagonals and each diagonal has the same length as one of its sides, or prove that such a polygon does not exist.

5. A palindrome is a number that remains the same when its digits are reversed. Let n be a product of distinct primes not divisible by 10. Prove that infinitely many multiples of n are palindromes.

6. Let $n \geq 2$ be a positive integer. Determine the number of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_k \in \{0, 1, 2\}$ for $1 \leq k \leq n$ and $\sum_{k=1}^{n} x_k - \prod_{k=1}^{n} x_k$ is divisible by 3.

7. Let n be a positive integer, with prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

for distinct primes $p_1, \ldots, p_r$, and e_i positive integers. Define

$$rad(n) = p_1 p_2 \cdots p_r,$$

the product of all distinct prime factors of n.

Find all polynomials $P(x)$ with rational coefficients such that there exist infinitely many positive integers n with $P(n) = rad(n)$.

8. Let n and k be positive integers with $1 \leq k \leq n$. A set of cards numbered 1 to n are arranged randomly in a row from left to right. A person alternates between performing the following moves:

   (a) The leftmost card in the row is moved $k - 1$ positions to the right while the cards in positions 2 through k are each moved one place to the left.

   (b) The rightmost card in the row is moved $k - 1$ positions to the left while the cards cards in positions through $n - k + 1$ through $n - 1$ are each moved one place to the right.

   Determine the probability that after some number of moves the cards end up in order from 1 to n, left to right.