

2012 Sun Life Financial Repêchage Solutions

1. The front row of a movie theatre contains 45 seats.
 - (a) If 42 people are sitting in the front row, prove that there are 10 consecutive seats that are all occupied.
 - (b) Show that this conclusion doesn't necessarily hold if only 41 people are sitting in the front row.
2. Given a positive integer m , let $d(m)$ be the number of positive divisors of m . Determine all positive integers n such that $d(n) + d(n + 1) = 5$.
3. We say that (a, b, c) form a *fantastic triplet* if a, b, c are positive integers, a, b, c form a geometric sequence, and $a, b + 1, c$ form an arithmetic sequence. For example, $(2, 4, 8)$ and $(8, 12, 18)$ are fantastic triplets. Prove that there exist infinitely many fantastic triplets.
4. Let ABC be a triangle such that $\angle BAC = 90^\circ$ and $AB < AC$. We divide the interior of the triangle into the following six regions:

$$\begin{aligned}
 S_1 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PA < PB < PC \\
 S_2 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PA < PC < PB \\
 S_3 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PB < PA < PC \\
 S_4 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PB < PC < PA \\
 S_5 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PC < PA < PB \\
 S_6 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PC < PB < PA.
 \end{aligned}$$

Suppose that the ratio of the area of the largest region to the area of the smallest non-empty region is $49 : 1$. Determine the ratio $AC : AB$.

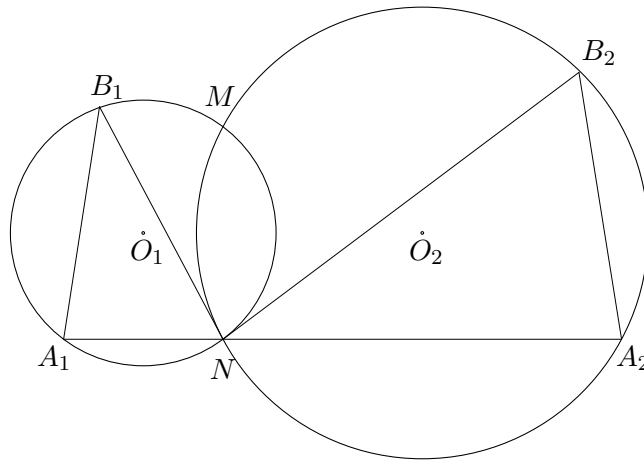
5. Given a positive integer n , let $d(n)$ be the largest positive divisor of n less than n . For example, $d(8) = 4$ and $d(13) = 1$. A sequence of positive integers a_1, a_2, \dots satisfies

$$a_{i+1} = a_i + d(a_i),$$

for all positive integers i . Prove that regardless of the choice of a_1 , there are infinitely many terms in the sequence divisible by 3^{2011} .

6. Determine whether there exist two real numbers a and b such that both $(x - a)^3 + (x - b)^2 + x$ and $(x - b)^3 + (x - a)^2 + x$ contain only real roots.
7. Six tennis players gather to play in a tournament where each pair of persons play one game, with one person declared the winner and the other person the loser. A triplet of three players $\{A, B, C\}$ is said to be *cyclic* if A wins against B , B wins against C and C wins against A .
 - (a) After the tournament, the six people are to be separated in two rooms such that none of the two rooms contains a cyclic triplet. Prove that this is always possible.

- (b) Suppose there are instead seven people in the tournament. Is it always possible that the seven people can be separated in two rooms such that none of the two rooms contains a cyclic triplet?
8. Suppose circles W_1 and W_2 , with centres O_1 and O_2 respectively, intersect at points M and N . Let the tangent on W_2 at point N intersect W_1 for the second time at B_1 . Similarly, let the tangent on W_1 at point N intersect W_2 for the second time at B_2 . Let A_1 be a point on W_1 which is on arc B_1N not containing M and suppose line A_1N intersects W_2 at point A_2 . Denote the incentres of triangles B_1A_1N and B_2A_2N by I_1 and I_2 , respectively.¹



Show that

$$\angle I_1 M I_2 = \angle O_1 M O_2.$$

¹Given a triangle ABC , the incentre of the triangle is defined to be the intersection of the angle bisectors of A , B and C . To avoid cluttering, the incentre is omitted in the provided diagram. Note also that the diagram serves only as an aid and is not necessarily drawn to scale.

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1. The front row of a movie theatre contains 45 seats.
 - (a) If 42 people are sitting in the front row, prove that there are 10 consecutive seats that are all occupied.
 - (b) Show that this conclusion doesn't necessarily hold if only 41 people are sitting in the front row.

Solution:

(a) We first number the seats in the row in order from 1 to 45. Suppose on the contrary that every group of 10 consecutive seats contains an unoccupied seat. We split the 45 seats into the following five groups.

$$\begin{aligned}S_1 &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\S_2 &= \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\} \\S_3 &= \{21, 22, 23, 24, 25, 26, 27, 28, 29, 30\} \\S_4 &= \{31, 32, 33, 34, 35, 36, 37, 38, 39, 40\} \\S_5 &= \{41, 42, 43, 44, 45\}\end{aligned}$$

Since each of S_1, S_2, S_3, S_4 consists of 10 consecutive seats, each of S_1, S_2, S_3, S_4 contains an unoccupied seat. This implies that there are at least 4 unoccupied seats. Hence, at most 41 seats are occupied. This contradicts that there are 42 people seated in the front row. Therefore, there are 10 consecutive seats that are all occupied. \square

(b) We give an example of how 41 people can be seated without ten consecutive seats being occupied. Equivalently, we give the location of $45 - 41 = 4$ empty seats such that every set of ten consecutive seats contains an empty seat. If we leave seats 10, 20, 30, 40 empty, then every group of ten consecutive seats contains an empty seat, since every set of ten consecutive numbers contains a seat whose right-most digit is 0. \square

2. Given a positive integer m , let $d(m)$ be the number of positive divisors of m . Determine all positive integers n such that $d(n) + d(n + 1) = 5$.

Solution: The answers are $n = 3$ and $n = 4$.

Since $d(m)$ is a positive integer for all positive integers m , we have $(d(n), d(n + 1)) = (1, 4), (2, 3), (3, 2)$ or $(4, 1)$. Note that the only positive integer m satisfying $d(m) = 1$ is $m = 1$. Therefore, if $d(n) = 1$, then $n = 1$. But $d(2) = 2 \neq 4$. Therefore, $d(n) \neq 1$. If $d(n + 1) = 1$, then $n + 1 = 1$. Hence, $n = 0$, which is not a positive integer. Therefore,

$$(d(n), d(n+1)) = (2, 3) \text{ or } (3, 2).$$

Note that $d(m) = 2$ if and only if m is prime. If $d(m) = 3$, then m is composite. However, m cannot be the product of two distinct integers $a, b \neq 1$, since then $1, a, b, m$ would all be positive divisors of m , implying $d(m) \geq 4$. The only positive integers m satisfying $d(m) = 3$ are squares of a prime number. Therefore, if $(d(n), d(n+1)) = (2, 3)$, then n is prime and $n+1$ is the square of a prime. Similarly, if $(d(n), d(n+1)) = (3, 2)$, then n is a square of a prime and $n+1$ is prime.

If $d(n) = 2$ and $d(n+1) = 3$, then $n = p$ and $n+1 = q^2$ for some primes p and q . This implies $p = q^2 - 1 = (q+1)(q-1)$. Since p is prime, $q-1 = 1$, i.e. $q = 2$. Thus, $n+1 = q^2 = 4$, i.e. $n = 3$. We check that $n = 3$ is a solution by noting that $d(3) + d(4) = 2 + 3 = 5$. If $d(n) = 3$ and $d(n+1) = 2$, then $n = p^2$ and $n+1 = q$ for some primes p and q . This implies that $q = p^2 + 1$. If p is odd, then $p \geq 3$. Hence, q is an even positive integer greater than or equal to $3^2 + 1 = 10$, implying that q is composite. This contradicts q being prime. Hence, p is even, i.e. $p = 2$. Then $n = p^2 = 4$. We check that $n = 4$ is a solution by noting that $d(4) + d(5) = 3 + 2 = 5$. So the only possible solutions are $n = 3$ and $n = 4$. \square

3. We say that (a, b, c) form a *fantastic triplet* if a, b, c are positive integers, a, b, c form a geometric sequence, and $a, b+1, c$ form an arithmetic sequence. For example, $(2, 4, 8)$ is a fantastic triplet. Prove that there exist infinitely many fantastic triplets.

Solution: Note that $(2, 4, 8)$ and $(8, 12, 18)$ are fantastic triplets. We will now show that $(2m^2, 2m(m+1), 2(m+1)^2)$ is a fantastic triplet for every positive integer m . This shows that there are infinitely many fantastic triplets. Note that this sequence has a common ratio of $(m+1)/m$. Therefore, $2m^2, 2m(m+1), 2(m+1)^2$ is a geometric sequence. Finally, $(2m^2, 2m(m+1)+1, 2(m+1)^2) = (2m^2, 2m^2+2m+1, 2m^2+4m+2)$ is an arithmetic sequence, with common difference $2m+1$. Therefore, $(2m^2, 2m(m+1), 2(m+1)^2)$ is a fantastic triplet. \square

Comment: The following is a proof that all fantastic triplets are of the form $(2m^2, 2m(m+1), 2(m+1)^2)$ or $(2(m+1)^2, 2m(m+1), 2m^2)$, where m is a positive integer.

Note that (a, b, c) is a fantastic triplet if and only if (c, b, a) is a fantastic triplet. Hence, we may assume that $a \leq b \leq c$.

Since a, b, c is a geometric sequence, $b = \sqrt{ac}$. Since $a, b+1, c$ is an arithmetic sequence,

$$b+1 = \frac{a+c}{2}.$$

Substituting $b = \sqrt{ac}$ into this equation yields

$$\begin{aligned} \sqrt{ac} + 1 &= \frac{a+c}{2} \\ \Rightarrow 2\sqrt{ac} + 2 &= a + c \\ \Rightarrow 2 &= a + c - 2\sqrt{ac} \\ \Rightarrow 2 &= (\sqrt{a} - \sqrt{c})^2. \end{aligned}$$

Since $a \leq c$, $\sqrt{c} - \sqrt{a} = \sqrt{2}$. We rewrite this as $\sqrt{c} = \sqrt{a} + \sqrt{2}$. Squaring both sides yields

$$c = a + 2 + 2\sqrt{2a}.$$

Since c is an integer and $2a$ is an integer, $2a$ is a perfect square. Since $2a$ is even, $2a$ is an even perfect square. Therefore, there exists a positive integer m such that $2a = (2m)^2$. Therefore, $a = 2m^2$. Hence, $\sqrt{c} - \sqrt{2m^2} = \sqrt{2}$, which implies that $\sqrt{c} = \sqrt{2}(1 + m)$. Therefore, $c = 2(m + 1)^2$. Hence, $b = \sqrt{ac} = 2m(m + 1)$.

Therefore, $(a, b, c) = (2m^2, 2m(m + 1), 2(m + 1)^2)$ for some positive integer m . We claim that this is a fantastic triplet for any positive integer m . This will show that there are infinitely many fantastic triplets. (In fact, we have shown that these are all of the fantastic triplets.) By construction of b , we have that a, b, c is a geometric sequence. It remains to show that $a, b + 1, c$ is an arithmetic sequence. Note that $a = 2m^2$, $b + 1 = 2m^2 + 2m + 1$ and $c = 2(m + 1)^2 = 2m^2 + 4m + 2$. Clearly, $a, b + 1, c$ is an arithmetic sequence (with common difference $2m + 1$). Therefore, $(a, b, c) = (2m^2, 2m(m + 1), 2(m + 1)^2)$ is a fantastic triplet for any positive integer m . \square

4. Let ABC be a triangle such that $\angle BAC = 90^\circ$ and $AB < AC$. We divide the interior of the triangle into the following six regions:

$$\begin{aligned} S_1 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PA < PB < PC \\ S_2 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PA < PC < PB \\ S_3 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PB < PA < PC \\ S_4 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PB < PC < PA \\ S_5 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PC < PA < PB \\ S_6 &= \text{set of all points } P \text{ inside } \triangle ABC \text{ such that } PC < PB < PA. \end{aligned}$$

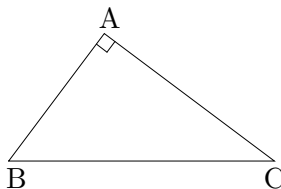
Suppose that the ratio of the area of the largest region to the area of the smallest non-empty region is $49 : 1$. Determine the ratio $AC : AB$.

Solution: We recall that given two distinct points M, N , let ℓ be the perpendicular bisector of MN , i.e. the line perpendicular to MN passing through the midpoint of MN . Then the set of all points P such that $PM < PN$ is the set of all points in the half-plane of ℓ containing M . Similarly, the set of all points P such that $PM > PN$ is the set of all points

in the half-plane of ℓ containing N . We will note this property by (*).

Back to the original problem. Let D, E, F be the midpoints of BC, CA, AB , respectively. Since $\angle BAC = 90^\circ$, DF is the perpendicular bisector of AB . Similarly, DE is the perpendicular bisector of AC . Consider the perpendicular bisector of BC ; this line passes through the point D . Since $AB < AC$, this perpendicular bisector passes through side AC , say at a point Q . We now characterize $S_1, S_2, S_3, S_4, S_5, S_6$.

By (*), S_1 is the quadrilateral $AFDQ$, S_2 is triangle QDE , S_3 is triangle BDF . S_4 is empty. S_5 is triangle DCE and S_6 is empty. Since S_4 and S_6 are empty, they are neither the largest region nor the smallest non-empty region.



Let $[X]$ be the area of a region X . Now let $AF = FB = 2$, $AE = EC = x$, and $EQ = y$. Then $[S_3] = [S_5] = x$, $[S_2] = y$, and $[S_1] = 2x - y$. Since $x > y$, S_1 must be the largest region and S_2 must be the smallest non-empty region, implying $(2x - y)/y = 49$, or $x = 25y$. Since $\triangle EDC$ is similar to $\triangle EQD$, $x/2 = 2/y$, which implies that $xy = 4$. Combining this with $x = 25y$ yields $25y^2 = 4$. Hence, $y = 2/5$. Therefore, $x = 10$. We conclude that $AC : AB = 2x : 4 = 20 : 4 = 5 : 1$. \square

5. Given a positive integer n , let $d(n)$ be the largest positive divisor of n less than n . For example, $d(8) = 4$ and $d(13) = 1$. A sequence of positive integers a_1, a_2, \dots satisfies

$$a_{i+1} = a_i + d(a_i),$$

for all positive integers i . Prove that regardless of the choice of a_1 , there are infinitely many terms in the sequence divisible by 3^{2011} .

Solution: For each positive integer m , let $f(m)$ be the largest positive integer t such that 3^t divides m . It suffices to show that there are infinitely many terms in the sequence satisfying $f(a_i) \geq 2011$. I claim that for any positive index r , there exists a positive index $s > r$ such that $f(a_s) > f(a_r)$. This claim will imply the result of the problem, since by this claim, eventually, there is a term in the sequence divisible by 3^{2011} . By this same claim, there is a term after this which is also divisible by 3^{2011} . Repeatedly applying this claim yields infinitely many terms in the sequence divisible by 3^{2011} .

We will consider two cases: when a_r is even and when a_r is odd.

If a_r is even, then $d(a_r) = a_r/2$. Therefore, $a_{r+1} = a_r + a_r/2 = 3a_r/2$. Then $f(a_{r+1}) = f(a_r) + 1$. By setting $s = r + 1$, we have proven the claim.

If a_r is odd, then let p be the smallest prime divisor of a_r . We will consider two subcases: when a_r is not divisible by 3 and when a_r is divisible by 3.

If a_r is not divisible by 3, then $p \geq 5$. Note that since a_r is odd, every divisor of a_r is odd. In particular, $d(a_r)$ is odd. Then $a_r + d(a_r)$ is even, i.e. a_{r+1} is even. Note also that $d(a_r) = a_r/p$. Therefore,

$$a_{r+1} = \frac{p+1}{p} \cdot a_r.$$

Since $p \geq 5$, $p \neq 3$. Hence, $f(a_{r+1}) \geq f(a_r)$. Since a_{r+1} is even, from the case where a_r is even, we have that $f(a_{r+2}) > f(a_{r+1}) \geq f(a_r)$. Hence, our claim is shown by setting $s = r + 2$.

Finally, if a_r is divisible by 3, then $p = 3$. Then

$$a_{r+1} = \frac{4}{3} \cdot a_r.$$

Therefore, $f(a_{r+1}) = f(a_r) - 1$. However, note that a_{r+1} is divisible by 4. Hence,

$$a_{r+2} = \frac{3}{2} \cdot a_{r+1}.$$

Therefore, $f(a_{r+2}) = f(a_{r+1}) + 1$. Hence, $f(a_{r+2}) = f(a_r)$. Since a_{r+1} is divisible by 4, a_{r+2} is divisible by 2. From the case where a_r is even, we have $f(a_{r+3}) > f(a_{r+2}) = f(a_r)$. Therefore, our claim is proved by setting $s = r + 3$.

This proves the problem statement. \square

6. Determine whether there exist two real numbers a and b such that both $(x - a)^3 + (x - b)^2 + x$ and $(x - b)^3 + (x - a)^2 + x$ contain only real roots.

Solution: The answer is no.

Suppose $(x - a)^3 + (x - b)^2 + x$ contain only real roots. Let r, s, t be real roots of $(x - a)^3 + (x - b)^2 + x = x^3 - (3a - 1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2)$. Then

$$(x - r)(x - s)(x - t) = x^3 - (3a - 1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2).$$

The left hand side simplifies to

$$x^3 - (r + s + t)x^2 + (rs + st + tr)x - rst.$$

Therefore, $r + s + t = 3a - 1$ and $rs + st + tr = 3a^2 - 2b + 1$. Note that

$$\begin{aligned} (r + s + t)^2 &= r^2 + s^2 + t^2 + 2(rs + st + tr) \\ &= \left(\frac{r^2+s^2}{2} + \frac{s^2+t^2}{2} + \frac{t^2+r^2}{2} \right) + 2(rs + st + tr) \\ &\geq (rs + st + tr) + 2(rs + st + tr) \\ &= 3(rs + st + tr). \end{aligned}$$

(Here, we use the inequality $x^2 + y^2 \geq 2xy$ for all real numbers x, y , which is true since this inequality is equivalent to $(x - y)^2 \geq 0$.) Therefore,

$$(3a - 1)^2 \geq 3(3a^2 - 2b + 1).$$

Equivalently, $6a - 6b \leq -2$. Similarly, since $(x - b)^3 + (x - a)^2 + x$ contain only real roots, $6b - 6a \leq -2$. Adding these two inequalities yields $0 \leq -4$, which is a contradiction. Therefore, both $(x - a)^3 + (x - b)^2 + x$ and $(x - b)^3 + (x - a)^2 + x$ cannot contain only real roots. \square

7. Six tennis players gather to play in a tournament where each pair of persons play one game, with one person declared the winner and the other person the loser. A triplet of three players $\{A, B, C\}$ is said to be *cyclic* if A wins against B , B wins against C and C wins against A .
- (a) After the tournament, the six people are to be separated in two rooms such that none of the two rooms contains a cyclic triplet. Prove that this is always possible.
 - (b) Suppose there are instead seven people in the tournament. Is it always possible that the seven people can be separated in two rooms such that none of the two rooms contains a cyclic triplet?

Solution:

Solution 1 to (a): Let A, B, C, D, E, F be the six tennis players and $w_1, w_2, w_3, w_4, w_5, w_6$ be the number of wins by A, B, C, D, E, F , respectively. Define a *dominating triplet* to be any triplet that is not a cyclic triplet, A dominating triplet has the property that one team in the triplet wins 2 games, one team wins 1 game, one team wins 0. For each dominating triplet, let the team who wins two games within the triplet be called the *dominator*.

For each team, we count the number of dominating triplets for which the team is the dominator. For any team T , let \mathcal{S} be the set of teams that T wins against. Note that every pair of teams in \mathcal{S} , in conjunction with T , form the set of all dominating triplets for which team T is the dominator. Therefore, if we let w be the number of wins that team T has, then the

number of dominating triplets for which team T is the dominator is $\binom{w}{2}$.

Hence, the number of dominating triplets where A, B, C, D, E, F are dominators is

$$\binom{w_1}{2}, \binom{w_2}{2}, \binom{w_3}{2}, \binom{w_4}{2}, \binom{w_5}{2}, \binom{w_6}{2},$$

respectively. Since every dominating triplet contains a unique dominator, the total number of dominating triplets is

$$\sum_{i=1}^6 \binom{w_i}{2} = \sum_{i=1}^6 \frac{w_i(w_i - 1)}{2} = \frac{1}{2} \cdot \sum_{i=1}^6 (w_i^2 - w_i) = \frac{1}{2} \cdot \left(\sum_{i=1}^6 w_i^2 - \sum_{i=1}^6 w_i \right).$$

Note that since there are $\binom{6}{2} = 15$ games are played, $w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 15$. Hence, the number of dominating triplets is

$$\frac{1}{2} \cdot \left(\left(\sum_{i=1}^6 w_i^2 \right) - 15 \right).$$

Note that by Cauchy-Schwarz inequality, we have

$$(w_1^2 + w_2^2 + \dots + w_6^2)(1 + 1 + \dots + 1) \geq (w_1 + w_2 + \dots + w_6)^2.$$

Hence,

$$\sum_{i=1}^6 w_i^2 \geq \frac{(w_1 + w_2 + w_3 + w_4 + w_5 + w_6)^2}{1 + 1 + 1 + 1 + 1 + 1} = \frac{15^2}{6} > 37.$$

Therefore, the number of dominating triplets is greater than

$$\frac{1}{2} \cdot (37 - 15) = 11.$$

Hence, the number of dominating triplets is at least 12. Since the number of dominating triplets equals to the number of triplets total minus the number of cyclic triplets and the former is equal to $\binom{6}{3} = 20$, there are at most 8 cyclic triplets. But there are ten ways to split a group of six people into two rooms of three people each. (This is shown by noting that there are $\binom{5}{2} = 10$ choices of two people that A can share a room with.) Since $8 < 10$, one of these groupings does not contain a cyclic triplet. This solves the problem. \square

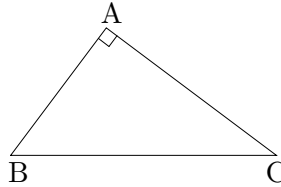
Solution 2 to (a): Let A, B, C, D, E, F be the six tennis players. Since $\binom{6}{2} = 15$ games are played and there are six players, there is one player that is the winner of $\lfloor 15/6 \rfloor + 1 = 3$ games. Without loss of generality, suppose A won three games, say against B, C, D . If $\{B, C, D\}$ is not cyclic, then $\{A, B, C, D\}$ does not contain a cyclic triplet. Clearly, $\{E, F\}$ does not

containing a cyclic triplet, since there are only two teams. Therefore, we can split the six people into the two groups $\{A, B, C, D\}$ and $\{E, F\}$, with neither group containing a cyclic triplet.

Otherwise, suppose $\{B, C, D\}$ is cyclic. By symmetry, suppose that B wins against C , C wins against D and D wins against B . Now, by symmetry, suppose E wins against F .

If $\{B, E, F\}$ is not cyclic, then $\{A, C, D\}$ and $\{B, E, F\}$ are both not cyclic, which results in two rooms not containing a cyclic triplet. The same results hold if any of $\{C, E, F\}$ and $\{D, E, F\}$ is not cyclic. Therefore, it remains to handle the case when $\{B, E, F\}$, $\{C, E, F\}$, $\{D, E, F\}$ are each cyclic. Since E wins against F , F wins against B, C and D and E loses against B, C and D . Then $\{A, B, F\}$ is not cyclic since both A, F won against B and $\{C, D, E\}$ is not cyclic, since C wins against both D and E . Hence, splitting the six people into the following two groups $\{A, B, F\}$ and $\{C, D, E\}$ would yield both rooms not containing a cyclic triplet. \square

(b) No, it is not always possible. Let A_0, A_1, \dots, A_6 be seven people. Now suppose A_i beats A_j if and only if $j - i \equiv 1, 2, 4 \pmod{7}$. Note that $\pm 1, \pm 2, \pm 4 \pmod{7}$ cover all six non-zero modulo classes $\pmod{7}$. Hence, this assignment of who wins in each game is consistent.

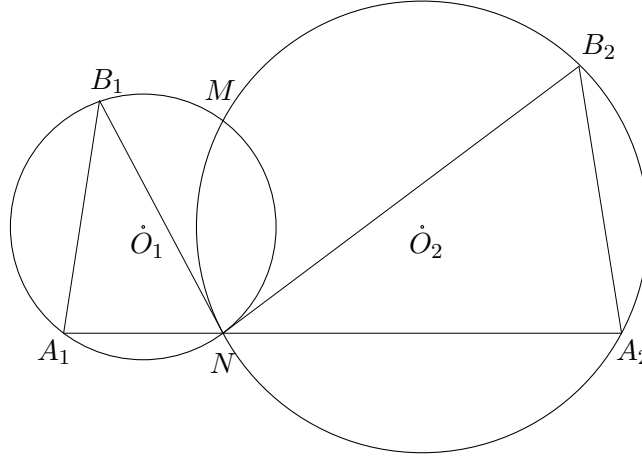


Since there are seven players, when splitting the seven people into two rooms, one room contains at least four people. I claim that a cyclic triplet exists in the room containing at least four people.

Amongst the four people are chosen from A_0, A_1, \dots, A_6 , then there are two people of the form A_i and A_{i+1} for some non-negative integer i . By cyclicity, suppose that A_0 and A_1 are among the four people in the room. Note that $\{A_0, A_1, A_3\}$ and $\{A_0, A_1, A_5\}$ are cyclic. Hence, if A_3 or A_5 is amongst the other two people in the room aside from A_0, A_1 , then the room contains a cyclic triplet. If A_2 is amongst the group of four, then note that $\{A_0, A_2, A_6\}$ and $\{A_1, A_2, A_4\}$ are both cyclic. The only remaining case is when the group of four is $\{A_0, A_1, A_4, A_6\}$. Note that $\{A_0, A_4, A_6\}$ is cyclic. Therefore, we have shown that regardless of which four people are chosen from A_0, \dots, A_6 , three of the four people form a cyclic triplet. \square

8. Suppose circles W_1 and W_2 , with centres O_1 and O_2 respectively, intersect at points M and N . Let the tangent on W_2 at point N intersect W_1 for the second time at B_1 . Similarly, let

the tangent on W_1 at point N intersect W_2 for the second time at B_2 . Let A_1 be a point on W_1 which is on arc B_1N not containing M and suppose line A_1N intersects W_2 at point A_2 . Denote the incentres of triangles B_1A_1N and B_2A_2N by I_1 and I_2 , respectively.²



Show that

$$\angle I_1 M I_2 = \angle O_1 M O_2.$$

Solution: Since B_1N is tangent to W_2 at N , $\angle B_1NO_2 = 90^\circ$. Note that $\angle O_2MN = \angle O_2NM$ in triangle ΔO_2MN . Hence, $\angle MNO_2 < 90^\circ = \angle B_1NO_2$. Therefore, B_1 is on arc MN of circle W_1 exterior of circle W_2 . Similarly, B_2 is on arc MN of circle W_2 exterior of circle W_1 . Since A_1 is on arc B_1N of circle W_1 not containing M , A_2 is on arc B_2N of circle W_2 not containing M .

By the tangent-chord property, $\angle B_1A_1N = \angle B_1NB_2 = \angle B_2A_2N$ and $\angle A_1B_1N = 180^\circ - \angle A_1NB_2 = \angle B_2NA_2$. Therefore, ΔB_1A_1N is similar to ΔNA_2B_2 . (1)

Again, by the tangent-chord property, $\angle MB_2N = \angle MNB_1$ and $\angle NMB_2 = 180^\circ - \angle B_1NB_2 = \angle B_1MN$. Therefore, $\Delta MB_1N \sim MNB_2$. (2)

Combining (1) and (2) yields that quadrilateral MB_1A_1N and quadrilateral MNA_2B_2 are similar. These two quadrilaterals are cyclic.

Since I_1 is the incentre of ΔB_1A_1N and I_2 is the incentre of ΔNA_2B_2 , by these similarities, we have that $\angle I_1MN = \angle I_2MB_2$. Therefore, $\angle I_1MI_2 = \angle I_1MN + \angle NMI_2 = \angle I_2MB_2 + \angle NMI_2 = \angle NMB_2$.

²Given a triangle ABC , the incentre of the triangle is defined to be the intersection of the angle bisectors of A , B and C . To avoid cluttering, the incentre is omitted in the provided diagram.

Since O_1, O_2 are the centres of MB_1A_1N and MNA_2B_2 , respectively, $\angle O_1MN = \angle O_2MB_2$. Therefore, $\angle O_1MO_2 = \angle O_1MN + \angle NMO_2 = \angle O_2MB_2 + \angle NMO_2 = \angle NMB_2$. Therefore, $\angle I_1MI_2 = \angle O_1MO_2$, as desired. \square