XXX Asian Pacific Mathematics Olympiad

March, 2018

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo.ommenlinea.org`*.*
Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.

Problem 1. Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Proposed by Mahdi Etesamifard, Iran

Problem 2. Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$$

and

$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}.$$

Prove that

$$|f(x) - g(x)| > 2$$

for any non-integer real number x satisfying $0 < x < 2018$.

Proposed by Senior Problems Committee of the Australian Mathematical Olympiad Committee

Problem 3. A collection of n squares on the plane is called *tri-connected* if the following criteria are satisfied:

(i) All the squares are congruent.

(ii) If two squares have a point P in common, then P is a vertex of each of the squares.

(iii) Each square touches exactly three other squares.

How many positive integers n are there with $2018 \leq n \leq 3018$, such that there exists a collection of n squares that is tri-connected?

Proposed by Senior Problems Committee of the Australian Mathematical Olympiad Committee

Problem 4. Let ABC be an equilateral triangle. From the vertex A we draw a ray towards the interior of the triangle such that the ray reaches one of the sides of the triangle. When the ray reaches a side, it then bounces off following the *law of reflection*, that is, if it arrives with a directed angle α, it leaves with a directed angle $180^\circ - \alpha$. After n bounces, the ray returns to A without ever landing on any of the other two vertices. Find all possible values of n.

Proposed by Daniel Perales and Jorge Garza, Mexico

Problem 5. Find all polynomials $P(x)$ with integer coefficients such that for all real numbers s and t, if $P(s)$ and $P(t)$ are both integers, then $P(st)$ is also an integer.

Proposed by William Ting-Wei Chao, Taiwan