XXIX Asian Pacific Mathematics Olympiad

March, 2017

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo.ommenlinea.org`*.*
Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.

Problem 1. We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

Proposed by Warut Suksompong, Thailand

Problem 2. Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Proposed by Equipo Nicaragua, Nicaragua

Problem 3. Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \ldots \geq a_k$ of positive integers for which $a_1 + \cdots + a_k = n$ and each $a_i + 1$ is a power of two ($i = 1, 2, \ldots, k$). Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \ldots \geq b_m$ of positive integers for which $b_1 + \cdots + b_m = n$ and each inequality $b_j \geq 2b_{j+1}$ holds ($j = 1, 2, \ldots, m-1$).

Prove that $A(n) = B(n)$ for every positive integer n.

Proposed by Senior Problems Committee of the Australian Mathematical Olympiad Committee

Problem 4. Call a rational number r *powerful* if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all *powerful*.

Proposed by Jeck Lim, Singapur

Problem 5. Let n be a positive integer. A pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if

$$|a_1b_1 + \cdots + a_nb_n| \leq 1.$$

Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

Proposed by Pakawut Jiradilok and Warut Suksompong, Thailand